HYDROLOGIC OPTICS

Volume V. Properties

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Joint Tsunami Research Effort
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The importance of light in the sea is apparent when it is recalled that solar radiation supplies most of the energy input to the ocean and supports its ecology through photosynthesis. The biological productivity of an acre of ocean has been estimated to be, on a worldwide average, comparable to that of an acre of land.... All these aspects of light in the sea can be treated by describing the optical nature of ocean water.

S. Q. DUNTLEY
Light in the Sea [78]

VOLUME V

Chapter 8

Models for Irradiance Fields

Chapter 11

The Universal Radiative Transport Equation

PREFACE

The major emphasis in this volume of *Hydrologic Optics* is on the optical properties of the sea that govern the penetration of *natural* light into its depths. (The optical properties of artificial light fields were discussed in Section 1.5.) Because seas and lakes are largely horizontally stratified, we begin our studies with a simple irradiance model for light fields which describes the downward and upward flow of radiant energy in terms of a pair of coupled ordinary differential equations. This system of equations goes back in essence to the modern originator of our subject, Arthur Schuster [279]. In one sense the solution of these equations would constitute a simple exercise in an elementary differential equations course; in another and more profound sense these equations hold the conceptual keys to the subject of radiative transfer in scattering-absorbing media. It is not the mathematical simplicity of these equations that we shall exploit, but rather their conceptual content. Thus, throughout Chapter 8, we have an example of how to mine an unexpectedly deep vein of physical ideas centering on a system of equations ((8) of 8.5) that could otherwise be easily solved, and then forgotten in a few minutes, by an impatient young physical oceanographer who has visions of sun-glittered curling breakers awaiting him at the ocean's shore.

In Chapter 9 we step carefully into deeper waters to search for and distinguish between those optical properties of the sea that are either inherent or apparent. We begin with the apparent properties given us by the Schuster two-flow equations of Chapter 8. Then in ever more comprehensive terms we generalize these concepts and study their behavior under varying lighting conditions and physical settings, reaching a general classification in the closing section of Chapter 9.

In Chapter 10 we return again to the stratified light field, so prevalent in oceanography and limnology, and explore in detail the special and interesting behavior of the light field and its attendant apparent optical properties at both small and great depths in the sea.

Ms. Louise F. Lembeck typed the camera ready manuscript and assisted in editorial matters. The camera ready manuscript is an unchanged version of an earlier draft written in 1965. Parts of the chapters have been used in various course and seminar lectures over the years at Scripps Institution of Oceanography, La Jolla; the Naval Postgraduate School at Monterey, California; and at the Hawaii Institute of Geophysics, University of Hawaii.

R.W.P.

Honolulu,
August 1975

CHAPTER 8

MODELS FOR IRRADIANCE FIELDS

8.0 Introduction

In this chapter we shall develop some simple models of light fields in natural optical media, models which have been found to be most useful in the study of radiative transfer in the seas and the atmosphere. These models are built around the concept of irradiance and derive their simplicity and utility directly from the simplicity and utility of irradiance itself. For irradiance is the concept which describes radiant flux per unit area on a surface, and as such utilizes a single number rather than an infinite set of numbers as in the case of the radiance distribution studied throughout Chapter 7. The introduction of irradiance fields into the study of natural light fields is also encouraged by the following convenient geometric structure of the light fields found in the atmosphere, the sea, lakes and other natural hydrosols: over relatively great distances in all directions within horizontal planes in the sea or air, the natural light fields are often found to vary essentially very little, so that an irradiance value at one point of a plane matches that at other quite remote points on the plane; the points being 'remote' in the sense that they are separated by distances vast compared with the attenuation lengths of the media. This latter feature is especially noted in the seas and other laterally extensive natural hydrosols. Because of this gentle almost imperceptible variation over such planes, the radiometric field over these planes may be characterized by the single irradiance magnitude common to all points of a given plane. As a consequence the description of the flow of radiant energy down into the depths of the sea or into the atmosphere can often be reduced to the description of an irradiance flow along any one of the straight lines normal to the family of horizontal planes comprising that portion of the air or sea of interest. In short, the appropriate introduction of an irradiance field into a plane parallel medium reduces the description of the radiative transfer problem within that medium to a one-dimensional problem.

One of the more fascinating features of irradiance fields in plane-parallel media, especially for the theoretically inclined investigator, is the fundamental similarity between the basic equations for irradiance fields

and those for the radiance distributions as studied, e.g., in Chapter 7. The similarity is a thorough-going one which may, indeed, be used as a heuristic guide in pursuing either subject matter, using the other as a base. In this connection, it may be noted that the original forms of the invariant imbedding relations and internal source relations studied in Chapter 7 were tentatively found in the irradiance context by means of informal scratch pad calculations; the simplicity of their derivation and resultant forms in that context encouraged the rigorous search for the associated full fledged operator equations for radiance sprinkled throughout that chapter.

We shall be guided in our discussions in the initial sections of this chapter (8.1, 8.2) and once again in Section 8.7 by the close conceptual connections between the functional equations for irradiance and radiance fields. In this way we can effect a smooth transition from the general formulations of Chapter 7 to the simpler settings of this chapter and at the same time gain some insight into the unity of the theory engendered by the invariance concepts. However, for the main sections of the chapter (8.3-8.6) the discussions will for the most part dwell on specific models which have been tried and found useful in the daily tasks of obtaining numerical estimates and rule-of-thumb algebraic approximations to the magnitudes of light fields, and the properties of their associated optical media.

Throughout this chapter we shall work with an arbitrary plane-parallel medium $X(a,b)$ and adopt the reference frame for $X(a,b)$, as defined in Section 2.4. Furthermore the steady state irradiances $H(z,\pm)$ defined in Section 2.4, along with their attendant radiometric concepts, will be adopted without further explanation. The medium $X(a,b)$ will be assumed free of internal sources unless specifically noted otherwise, and arbitrarily stratified with a stratified light field, so that both radiometric and optical properties depend only on depth z within $X(a,b)$, $a \le z \le b$. Arbitrary sources are incident on the upper and lower boundaries of $X(a,b)$. (In real media, however, it is customary in practice to have no sources incident on the lower boundary.) The explicit retention of a source on the lower boundary will have the effect of keeping the resultant theoretical imbeding relations in their full symmetric form.

8.1 Invariant Imbedding Relation for Irradiance Fields

Our point of departure for the present discussion is the set of principles of invariance (7), (8) of Section 3.7. We recall that these statements were deduced from an application of the interaction principle to an arbitrary subslab $X(x,z)$ of a plane parallel medium of the type $X(a,b)$, schematically depicted in Fig. 8.1. The results may be written:

$$\text{I.} \quad H(y,+) = H(z,+)\,T(z,y) + H(y,-)\,R(y,z) \tag{1}$$

$$\text{II.} \quad H(y,-) = H(x,-)\,T(x,y) + H(y,+)\,R(y,x) \tag{2}$$

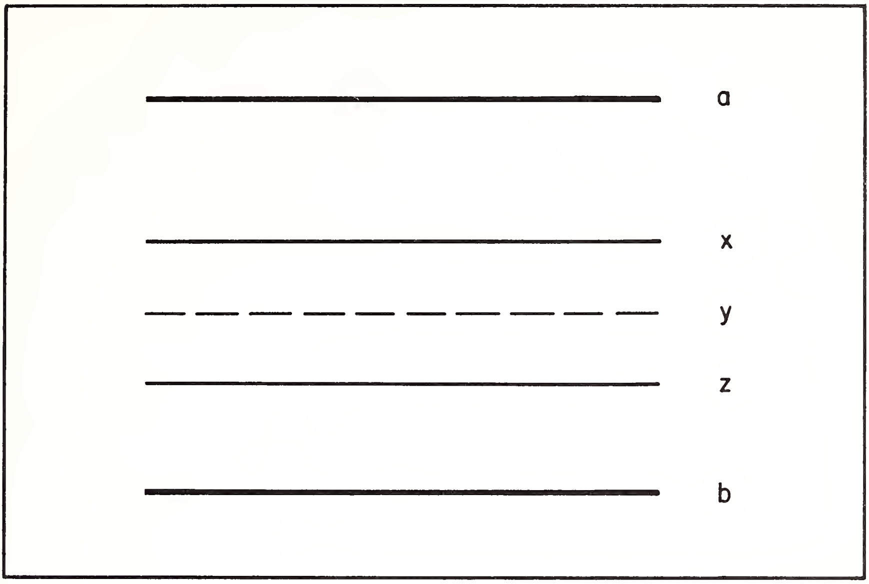

FIG. 8.1 The setting for the principles of invariance governing irradiance fields on plane-parallel media.

where $a \le x \le y \le z \le b$. The four *numbers* $T(x,y)$, $R(y,x)$, $T(z,y)$, $R(y,z)$ are the various *transmittances* (T) and *reflectances* (R) of the pieces $X(x,y)$ and $X(y,z)$ of the partitioned slab $X(x,z)$. The present goal is to derive the invariant imbedding relation for $X(a,b)$ in the irradiance context. Our activity will parallel very closely that in examples 4 and 5 of Section 3.9, thereby casting light on those earlier computations and in turn adding evidence to the belief that the manner of approach to the invariant imbedding relation, at least on the algebraic level, is independent of the geometry of the medium and the radiometric concepts used in the approach.

Most of the work toward attaining the invariant imbedding relation is already contained in the results (9), and (10) of Section 3.17; for if we now apply those equations to the subslab $X(x,z)$ of the present setting (by letting $z = x$, $b = z$) and write:

$$"\mathcal{R}(x,y,z)" \quad \text{for} \quad \frac{T(x,y)R(y,z)}{1-R(y,x)R(y,z)} \tag{3}$$

$$"\mathcal{T}(x,y,z)" \quad \text{for} \quad \frac{T(x,y)}{1-R(y,x)R(y,z)} \tag{4}$$

$$"\mathcal{R}(z,y,x)" \quad \text{for} \quad \frac{T(z,y)R(y,x)}{1-R(y,x)R(y,z)} \tag{5}$$

$$\text{"}\mathcal{T}(z,y,x)\text{"} \quad \text{for} \quad \frac{T(z,y)}{1-R(y,x)R(y,z)} \quad , \tag{6}$$

then those equations can be written:

$$H(y,+) = H(z,+)\mathcal{T}(z,y,x) + H(x,-)\mathcal{R}(x,y,z) \tag{7}$$

$$H(y,-) = H(x,-)\mathcal{T}(x,y,z) + H(z,+)\mathcal{R}(z,y,x) \tag{8}$$

Equations (7) and (8) present an excellent opportunity for the reader to become acquainted with the invariant imbedding relation in a relatively simple setting. It was for this reason that the irradiance example was presented first in Chapter 3. In this chapter we shall have more opportunity to explore the irradiance context of the invariant imbedding concepts.

The operators $\mathcal{R}$ and $\mathcal{T}$ defined in (3) through (6) above are the *complete reflectance* and *complete transmittance* factors, respectively. Observe that the following special cases of $\mathcal{R}$ and $\mathcal{T}$ hold:

$$\mathcal{R}(x,x,y) = R(x,y) \tag{9}$$

$$\mathcal{T}(x,y,y) = T(x,y) \tag{10}$$

$$\mathcal{R}(x,y,y) = 0 \tag{11}$$

$$\mathcal{T}(x,x,y) = 1 \tag{12}$$

These statements may be obtained directly from (3) and (4) upon suitable substitutions, and by appeal to (13) and (14) of Sec. 3.7. A complementary set of four equations can be obtained from (5) and (6). It follows that the invariant imbedding equations (7) and (8) contain the principles of invariance (1) and (2) as special cases. The equations (7) and (8) can be cast into matrix form by writing:

$$\text{"}\mathcal{M}(x,y,z)\text{"} \quad \text{for} \quad \begin{bmatrix} \mathcal{T}(z,y,x) & \mathcal{R}(z,y,x) \\ \mathcal{R}(x,y,z) & \mathcal{T}(x,y,z) \end{bmatrix} \tag{13}$$

so that we have:

$$\boxed{(H(y,+),H(y,-)) = (H(z,x),H(x,-))\mathcal{M}(x,y,z)} \tag{14}$$

This is the required *invariant imbedding relation* for the irradiance context. It should be observed that we are adopting in the irradiance context, without essential change, the notation for standard and complete operators used earlier in Chapter 3 for the radiance (operator) context. This affords a great economy of terminology, retains a useful and suggestive notation, and serves to strengthen the conceptual unity of radiative transfer theory attained by means of the invariant imbedding techniques. There shall be no confusion arising from this practice, for it very rarely happens that the

irradiance field and the radiance field are simultaneously under study in a given one-parameter medium, since these are two very distinct levels of description of given radiative transfer phenomena: the irradiance description presently under study is a simple numerical description of a light field while the radiance description is a more detailed functional description of the light field.

As it stands, the invariant imbedding relation (14) is the general form of the solution to the radiative transfer problem in $X(a,b)$ for the irradiance field: knowledge of $\mathcal{M}(x,y,z)$ for every three successive levels x,y,z in a subslab $X(x,z)$ of $X(z,b)$ allows one to compute the irradiance field $(H(y,+),H(y,-))$ at every level y in $X(x,z)$ knowing the incident radiance field $(H(z,+),H(x,-))$ on $X(x,z)$. The complete reflectance and transmittance operators $\mathcal{R}$ and $\mathcal{T}$ in $\mathcal{M}(x,y,z)$ depend in turn on the standard operators R and T as shown in (3) through (6). Therefore the complete solution of the irradiance transfer problem in $X(a,b)$ devolves on knowledge of the standard R and T *factors*, in exact analogy to the radiance transfer problem studied in detail in Chapter 7. Consequently, knowing the R and T, or better still the $\mathcal{R}$ and $\mathcal{T}$ factors, for a medium $X(a,b)$, we can write down by sight the answer to every question about $H(y,\pm)$ for every level y in $X(a,b)$. We shall in the course of this chapter obtain methods for the determination of the standard R and T factors and the complete factors $\mathcal{R}$ and $\mathcal{T}$. For the present we go on to formulate further equations governing the irradiance field.

8.2 General Irradiance Equations

The global description of the irradiance field in a plane parallel medium $X(a,b)$ as given by the invariant imbedding relation (14) of Sec. 8.1 will now be supplemented by a local description in the form of a pair of differential equations for the irradiances $H(z,\pm)$ as a function of depth z in $X(a,b)$. The approach we shall take at present is through the principles of invariance (1) and (2) of Sec. 8.1.

The idea of the derivation is quite simple: We isolate for attention the subslab $X(x,z)$ of $X(a,b)$ and form the difference quotient:

$$\frac{H(z,-) - H(x,-)}{z - x} \tag{1}$$

We then let x approach z and determine the associated limit of (1). The physical meaning of this activity should be carefully noted at the outset, as it will repeatedly suggest the subsequent moves in the sequence of explorations below. Thus (1) is the average rate of change of the downward irradiance field over the depth interval from x to z. As $z - x$ is made smaller and smaller (and hence $X(x,z)$ thinner and thinner) we increasingly localize the factors governing this average rate of change until, in the limit, we should have a completely local description of the change of the downward irradiance field at depth z.

Following the program just outlined, we set $y = z$ in (2) of Sec. 8.1, the result being:

$$H(z,-) = H(x,-)T(x,z) + H(z,+)R(z,x) \quad .$$

This representation of $H(z,-)$ is used in (1) to obtain:

$$\frac{H(z,-) - H(x,-)}{z - x} = H(x,-) \frac{[T(x,z) - 1]}{z - x} + H(z,+) \frac{R(z,x)}{z - x} \tag{2}$$

Now as x approaches z, the difference $z - x$ approaches zero, and the slab $X(x,z)$ becomes increasingly thinner, so that its downward transmittance $T(x,z)$ approaches 1 and its upward reflectance $R(z,x)$ approaches zero (cf. (9)-(12) of Sec. 7.3). Therefore the quotients on the right side of (2) have a chance of going to well-defined limits. Indeed, our discussion in Sec. 7.3 (see e.g., (9)-(12) of that section) prepares the ground for the following definitions; we write:

$$\text{"}\tau(z,-)\text{"} \quad \text{for} \quad \lim_{x \to z} \frac{T(x,z) - 1}{z - x} \tag{3}$$

$$\text{"}\rho(z,+)\text{"} \quad \text{for} \quad \lim_{x \to z} \frac{R(z,x)}{z - x} \tag{4}$$

Then, if we write as usual:

$$\frac{\text{"}dH(z,-)\text{"}}{dz} \quad \text{for} \quad \lim_{x \to z} \frac{H(z,-) - H(x,-)}{z - x} \tag{5}$$

equation (2) yields:

$$\boxed{\frac{dH(z,-)}{dz} = \tau(z,-)H(z,-) + \rho(z,+)H(z,+)} \tag{6}$$

This is the equation governing the downward irradiance field $H(z,-)$. In a similar way, setting $y = x$ in (1) of Sec. 8.1, forming the difference quotient:

$$-\frac{H(x,+) - H(z,+)}{(x - z)} = H(z,+) \left[\frac{T(z,x) - 1}{z - x}\right] + H(x,-) \frac{R(x,z)}{z - x}$$

and writing:

$$\text{"}\tau(z,+)\text{"} \quad \text{for} \quad \lim_{x \to z} \frac{T(z,x) - 1}{z - x}$$

$$\text{"}\rho(z,-)\text{"} \quad \text{for} \quad \lim_{x \to z} \frac{R(x,z)}{z - x}$$

and

$$\frac{\text{"}dH(z,+)\text{"}}{dz} \quad \text{for} \quad \lim_{x \to z} \frac{H(x,+) - H(z,+)}{x - z} \quad ,$$

the preceding equation yields:

$$\boxed{-\frac{dH(z,+)}{dz} = \tau(z,+)H(z,+) + \rho(z,-)H(z,-)} \tag{7}$$

This is the equation governing the upward irradiance field $H(z,+)$.

The notation in (6) and (7) is designed to point up the fundamental similarity of these equations to the principles of invariance (1) and (2) of Sec. 8.1. On the basis of this similarity (6) and (7) are also called the *local forms* of the principles of invariance, and $\tau(z,\pm)$ and $\rho(z,\pm)$ are the *local transmittance* and *local reflectance factors*, respectively for upward (+) and downward (-) irradiance. The unity of the invariant imbedding approach is underscored when (6) and (7) are compared with (5) and (6) of Sec. 7.1. In fact on the basis of this comparison, we are moved to write:

$$\text{"}H(z)\text{"} \quad \text{for} \quad (H(z,+),H(z,-)) \tag{8}$$

$$\text{"}\mathcal{K}(z)\text{"} \quad \text{for} \begin{bmatrix} -\tau(z,+) & \rho(z,+) \\ -\rho(z,-) & \tau(z,-) \end{bmatrix} \tag{9}$$

so that (6) and (7) can be written:

$$\boxed{\frac{dH(z)}{dz} = H(z)\mathcal{K}(z)} \tag{10}$$

Equation (10) is the *vector form* of the general irradiance equations, and is the irradiance counterpart to (9) of Sec. 7.1.

The practical distinction between (10) above and (9) of Sec. 7.1 should be kept firmly in mind: (10) is a vector equation whose components are *numbers* while (9) of Sec. 7.1 is a vector equation whose components are *functions* and thus one level higher in the conceptual hierarchy. However, both the functional and numerical components obey many of the same algebraic relations and, generally speaking, *whenever a functional equation deduced from (9) of Sec. 7.1 is valid, then there exists a corresponding valid counterpart deducible from (10).* It is almost as if the local forms of the theorems of irradiance fields are the one dimensional shadows of the corresponding theorems of radiance fields; similarly for deductions from the *global* forms of the principles of invariance and their irradiance correspondents. Caution should be exercised in attempting to extend results the other way, i.e., from the irradiance level (10) to the radiance level (9) of Sec. 7.1, or from (1) and (2) of Sec. 8.1 back to the principles in example 3 of Sec. 3.7. While no universal rule exists to guide extensions from the irradiance to the radiance level,

it is clear that if essential use is made of the commutativity of the *numerical factors* R and T while gaining a result, then the associated result need not exist on the radiance level, since commutativity of the R and T *operators* does not hold in general. Some examples using this observation were discussed in Sec. 7.13 (see (91) of Sec. 7.14).

We shall turn to the applications of (10) in the discussions of Sec. 8.7; for the present we continue to explore its analytic structure. Before going on to do so, we pause and note one rather interesting similarity between (10) above and the fundamental dynamical equation of quantum mechanics:

$$i\hbar \frac{d\,|\,\psi>}{dt} = \,|\,\psi > \hat{H} \tag{10a}$$

where $|\,\psi>$ is a state vector which pairs with our $H(z)$ and $\hat{H}$ (or $-(i/\hbar)\hat{H}$) is the Hamiltonian matrix operator which pairs with our $\mathcal{H}(z)$. As a result of this pairing we see that the mathematics of time-dependent atomic systems is homomorphic to (i.e., of the same kind as) that for steady irradiance fields in stratified media (cf. also (46) of Sec. 8.6 and the remarks following (91) of Sec. 3.7). One more connection between (10) above (and its generalization (46) of Sec. 8.6) and the mathematical structure of different fields of physics may be noted. This concerns the formulations by Brillouin* of the transmission line equations for two phase and polyphase electric fields. The use of Pauli and Dirac matrices to compactly represent circuit equations for such fields can evidently be carried over with only slight modifications to the radiative transfer context. However, in the present work we shall develop the algebra of radiative transfer on the basis of the invariant imbedding point of view introduced in Chapter 7.

8.3 Two-Flow Equations: Undecomposed Form

We now add another block to the foundations for the model constructions of irradiance fields to be given below by deriving the classical two-flow equations for irradiance which correspond to (6) and (7) of Sec. 8.2. The primary distinction between (6) and (7) above and the two-flow equations below lies in the structure of the coefficients of $H(z,\pm)$ in the respective equations. In order to arrive at the two-flow equations we shall analyze the local transmittance factors $\tau(z,\pm)$ and the local reflectance factors $\rho(z,\pm)$ into further parts and relate these parts directly to the radiance distributions and the inherent optical properties α, σ of the optical medium $X(a,b)$. In this way we will be able to make direct contact with certain well-known models of irradiance fields starting with the Schuster progenitors of classical radiative transfer theory, down through the variations wrought by Ryde, Gurevic, Duntley and others during the decades that followed. The historical details of

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*Brillouin, L., *Wave Propagation in Periodic Structures*. Dover Publications, New York (1953).

the manifold forms of the two-flow equations are reserved for discussion in the bibliographic notes appended to this chapter. For the present we go on to a modern development and analysis of the two-flow equations.

Our starting point for the present derivations is the steady state equation of transfer for radiance in a source-free isotropic stratified optical medium, i.e., we begin with (3) of Sec. 3.15 in the form:*

$$\xi \cdot \nabla N(z,\xi) = -\alpha(z)N(z,\xi) + \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad . \tag{1}$$

Here "z" denotes depth in the present stratified plane parallel medium $X(a,b)$, and the term "$\xi \cdot \nabla$" is defined in (6) of Sec. 3.15. If ever sources are to be taken into account, one need only add "$N_\eta(z,\xi)$" to the right side of (1). The net result on all subsequent equations is the addition of irradiance terms $H_\eta(z,\pm)$ to the right sides of the respective equations. The time-dependent version of (1) is obtained, as usual, by adding the time derivative term $(1/v)\,\partial N/\partial t$ to the left side.

The next step of the derivation is to integrate the terms of each side of (1) over the set Ξ_+ of upward directions. Taking the terms one by one, we begin with the derivative term:

$$\int_{\Xi_+} \xi \cdot \nabla N(z,)\,d\Omega(\xi) = -\int_{\Xi_+} \xi \cdot \mathbf{k}\,\frac{d}{dz}\,N(z,\xi)\,d\Omega(\xi)$$

$$= -\frac{d}{dz}\int_{\Xi_+} \xi \cdot \mathbf{k}N(z,\xi)\,d\Omega(\xi) = -\frac{dH(z,+)}{dz} \tag{2}$$

The first equality rests on the form of ∇ in a Cartesian coordinate frame in which z is measured positive in the direction $-\mathbf{k}$, as is the case in the terrestrial reference frame for hydrologic optics presently in use.** The stratified light field condition is also used to obtain the first equality, for under that condition the x and y derivatives of N vanish. The last equality is based on (9) of Sec. 2.4 and (8) of Sec. 2.5.

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*The isotropic assumption plays no essential role in this derivation in the sense that the structure of the resultant formulas (9) and (10) below are the same for the anisotropic case.

**If ∇ is to be used in a coordinate frame other than Cartesian then, in general (2) yields $\nabla \cdot \mathbf{H}(x,+)$, where $\mathbf{H}(x,+)$ is that contribution to $\mathbf{H}(x)$ by radiances in the directions of Ξ_+. See [221].

Starting now on the right side of (1), we integrate the linear term over Ξ_+ :

$$\int_{\Xi_+} \alpha(z)N(z,\xi)\,d\Omega(\xi) = \alpha(z)h(z,+) \tag{3}$$

in which we have used (7) and (11) of Sec. 2.7. Next, the integrated integral term in (1) becomes:

$$\int_{\Xi_+} N_*(z,\xi)\,d\Omega(\xi) =$$

$$= \int_{\Xi_+} \left[\int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi)$$

$$= \int_{\Xi_+} \left[\int_{\Xi_+} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right.$$

$$\left. + \int_{\Xi_-} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{4}$$

In the last equality, we have merely split the integration over Ξ into two parts: over Ξ_+ and over Ξ_- .

Equations (2), (3), and (4) are as far as we can go, blindly and mechanically. The next step in the derivations of the two-flow equations requires a strong sense of direction of the goal, namely two differential equations for the irradiances $H(z,\pm)$. Now, equation (2) shows us we are on the right track; but equation (3) shows us that we must perform some analytic legerdemain in order to obtain $H(z,+)$ from $h(z,+)$; and equation (4) shows us that the requisite analytic trickery must be thoroughgoing and boldly done. Some experimentation shows that we may profitably write:

$$\text{"}D(z,\pm)\text{"} \quad \text{for} \quad \frac{h(z,\pm)}{H(z,\pm)} \tag{5}$$

$$\text{"}\alpha(z,\pm)\text{"} \quad \text{for} \quad \alpha(z)D(z,\pm) \tag{6}$$

and:

$$\text{"}f(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H(z,\pm)} \int_{\Xi_\pm} \left[\int_{\Xi_\pm} N(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{7}$$

$$\text{"}b(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H(z,\pm)} \int_{\Xi_\mp} \left[\int_{\Xi_\pm} N(z,\xi')\sigma(z,\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{8}$$

It follows at once that (3) becomes:

$$\int_{\Xi_+} \alpha(z)N(z,\xi)\,d\Omega(\xi) = \alpha(z,+)H(z,+)$$

and (4) becomes:

$$\int_{\Xi_+} N_*(z,\xi)\,d\Omega(\xi) = f(z,+)H(z,+) + b(z,-)H(z,-) \quad ,$$

so that with (2), these results assemble into the requisite equation for $H(z,+)$:

$$\boxed{-\frac{dH(z,+)}{dz} = [f(z,+) - \alpha(z,+)]H(z,+) + b(z,-)H(z,-)} \tag{9}$$

Integrating (1) over Ξ_- , and using similar tactics to those described, we have:

$$\boxed{\frac{dH(z,-)}{dz} = [f(z,-) - \alpha(z,-)]H(z,-) + b(z,+)H(z,+)} \tag{10}$$

Equations (9) and (10) are the requisite two-flow equations for the irradiance fields $H(z,\pm)$. Comparison with (6) and (7) of Sec. 8.2 yields the important connections:

$$\tau(z,\pm) = f(z,\pm) - \alpha(z,\pm) \tag{11}$$

$$\rho(z,\pm) = b(z,\pm) \tag{12}$$

$f(z,\pm)$ and $b(z,\pm)$ are respectively, the *forward* and *backward scattering functions* for the irradiances $H(z,\pm)$; $\alpha(z,\pm)$ are the *attenuation functions* for $H(z,\pm)$, respectively. The functions $D(z,\pm)$ are called the *distribution functions* for $H(z,\pm)$. We also note the interesting connection between the values $\alpha(z,\pm)$ and those of $a(z,\pm)$ and $s(z,\pm)$ where we have written:

$$\text{"}a(z,\pm)\text{"} \quad \text{for} \quad a(z)D(z,\pm) \tag{13}$$

$$\text{"}s(z,\pm)\text{"} \quad \text{for} \quad s(z)D(z,\pm) \quad . \tag{14}$$

The connection of interest is:

$$\alpha(z,\pm) = a(z,\pm) + s(z,\pm) \quad , \tag{15}$$

which follows at once from the definition of the volume absorption function $a(z,\xi)$ given in (4) of Sec. 4.2; $a(z,\pm)$ and $s(z,\pm)$ are, respectively the *absorption* and *total scattering functions* for the irradiances $H(z,\pm)$. Equation (15) parallels the basic connection:

$$\alpha(z) = a(z) + s(z) \tag{16}$$

among the volume attenuation, absorption and total scattering functions in $X(a,b)$. Furthermore, from (3) of Sec. 4.2 and (7) and (8) above we have:

$$s(z,\pm) = f(z,\pm) + b(z,\pm) \tag{17}$$

which, combined with (15) yields:

$$\boxed{\alpha(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm)} \tag{18}$$

Equation (18) shows that the attenuation function for $H(z,\pm)$ is generally the sum of three terms: the absorption, forward and backward scattering functions. Using this connection, the two-flow equations (9) and (10) may be cast into their alternate forms:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [a(z,\pm) + b(z,\pm)]H(z,\pm) + b(z,\mp)H(z,\mp)} \tag{19}$$

We pause to examine the meanings of the terms in (19) and to sample the strong intuitive flavor of the two-flow equations. Choosing the upper signs in (19), we have the differential equation for $H(z,+)$ which states that the rate of change of the upward flow of radiant energy per unit area consists of three terms representing the simultaneous activity of the following three processes in $X(a,b)$:

(i) The decrease of $H(z,+)$ by absorption of $H(z,+)$ per unit length of travel.

(ii) The decrease of $H(z,+)$ by backscattering of $H(z,+)$ per unit length of travel.

(iii) The increase of $H(z,+)$ by backscattering of $H(z,-)$ per unit length of travel.

A similar interpretation may be assigned to the downward irradiance field $H(z,-)$ by replacing "+" by "-" throughout (i)-(iii) above. The minus sign before the derivative of $H(z,+)$ adjusts the vertical measurements to be positive upward for that equation, and is the vestige of the general convention to measure r positive in the direction ξ in the general equation of transfer (3) of Sec. 3.15.

Equilibrium Form of the Two-Flow Equations

We can cast the two-flow equations (19) into a form which points up even more strikingly the intuitive features of the irradiance field and which underscores still further their similarities to the radiance equations. We have in mind the introduction of the irradiance counterparts to the equilibrium form of the equation of transfer (4) of Sec. 4.3. Toward this end let us write:

$$\text{"}H_q(z,\pm)\text{"} \quad \text{for} \quad \frac{b(z,\mp)H(z,\mp)}{a(z,\pm) + b(z,\pm)} \tag{20}$$

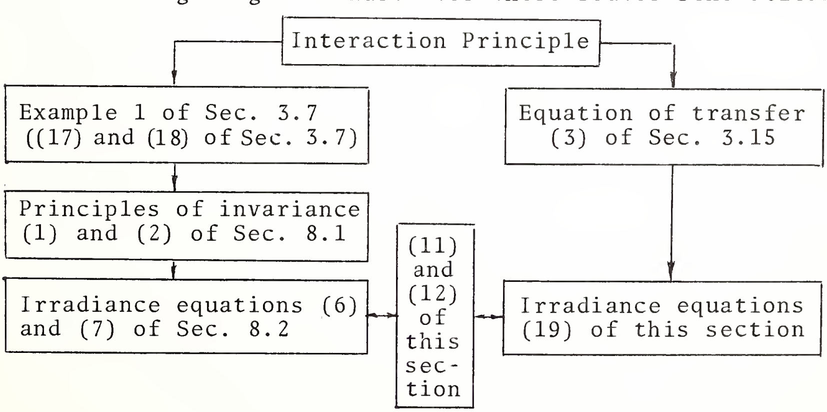
so that (19) becomes:

$$\mp \frac{dH(z,\pm)}{dz} = - [a(z,\pm) + b(z,\pm)] \, [H(z,\pm) - H_q(z,\pm)] \tag{21}$$

$H_q(z,\pm)$ is the *equilibrium irradiance* for $H(z,\pm)$. The reason for this nomenclature is obvious on inspection of (21). Consider $H(z,-)$: If $H(z,-) < H_q(z,-)$, then the derivative of $H(z,-)$ is positive and $H(z,-)$ is increasing toward $H_q(z,+)$. On the other hand, if $H_q(z,-) < H(z,-)$, then the derivative of $H(z,-)$ is negative and $H(z,-)$ is decreasing toward $H_q(z,-)$. Thus $H(z,-)$ relentlessly pursues $H_q(z,-)$. Unlike the race between N and N_q along a horizontal line of sight, that between $H(z,-)$ and $H_q(z,-)$ is never finished in real hydrosols. Reasons for this will become clear during our discussions in Chapters 9, 10, and 11.

Ontogeny of the Two-Flow Equations

Before going on to further derivations and to the solution procedures for the two-flow equations we pause to take note, for the student of radiative transfer theory, of the ontogenetic features of the two-flow equations. Historically, they trace back to Schuster's basic paper of 1905 (Ref. [279]). Logically, they rest on the interaction principle of Chapter 3. The logical route to them may be traversed in two distinct ways. The following diagram illustrates these routes schematically:



The bridge between the two forms of the irradiance equations is the pair of relations (11) and (12) which can be rigorously proved from their definitions and the interaction principle. However the simple visual match between the set (9) and (10) and (6) and (7) of Sec. 8.2 which suggests (11) and (12) will suffice for the purposes of this work. Interested students may attempt the direct proof of (11) and (12). It should be noted that the rigorous proof is not trivial and, if successfully done, has important related results in alternate modes of approach to radiative transfer theory (e.g., see step seven in Sec. 126 of Ref. [251]).

8.4 Two-Flow Equations: Decomposed Form

In this section we retrace the main steps of the preceding section with the goal in mind of deriving the two-flow equations for the decomposed light field in $X(a,b)$. The immediate basis for the derivation rests in (7) of Sec. 5.2. See also (19) through (22) of Sec. 5.1 wherein are also defined the notions underlying the idea of a decomposed light field. A suitable prerequisite for the present derivations are the discussions between (1) and (7) of Sec. 5.2, and between (56) and (62) of Sec. 6.6. The ultimate basis of the present discussion is (5) of Sec. 3.13.

Starting with (7) of Sec. 5.2:

$$\xi \cdot \nabla N^*(z,\xi) = -\alpha(z)H^*(z,\xi) + \int_{\Xi} N^*(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') + N^1_*(z,\xi) \tag{1}$$

where

$$N^1_*(z,\xi) = \int_{\Xi} N^\circ(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad , \tag{2}$$

we integrate each side of (1) over Ξ_+. The derivative term becomes:

$$\int_{\Xi_+} \xi \cdot \nabla N^*(z,\xi)\,d\Omega(\xi) = -\frac{d}{dz}\int_{\Xi_+} \xi \cdot \mathbf{k} N^*(z,\xi)\,d\Omega(\xi) \quad .$$

This motivates us to write:

$$\text{"}H^*(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi \cdot \mathbf{k}\,|\,N^*(z,\xi)\,d\Omega(\xi) \tag{3}$$

We call $H^*(z,\pm)$ the *diffuse upward (+) or downward (-) irradiance*. Similarly for $a \le z \le b$, we write:

$$\text{"}H^\circ(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_\pm} |\,\xi \cdot \mathbf{k}\,|\,N^\circ(z,\xi)\,d\Omega(\xi) \quad , \tag{4}$$

so that by (5) of Sec. 3.13:

$$\boxed{H(z,\pm) = H^{o}(z,\pm) + H^{*}(z,\pm)} \qquad (5)$$

We call $H^{o}(z,\pm)$ the *residual* (or reduced) *upward* (+) or *downward* (-) irradiance. Equation (5) exhibits the *decomposition of the irradiance fields.* The definitions (5) through (8) of Sec. 8.3 can now be repeated for the *diffuse* and *residual* irradiances by the simple expedient of placing a star (*) or zero (o) superscript on the radiometric quantities involved. For example in the case of diffuse irradiance we write:

$$\text{"}D^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{h^{*}(z,\pm)}{H^{*}(z,\pm)} \qquad (6)$$

$$\text{"}\alpha^{*}(z,\pm)\text{"} \quad \text{for} \quad \alpha(z)D^{*}(z,\pm) \qquad (7)$$

$$\text{"}f^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H^{*}(z,\pm)} \int_{\Xi_{\pm}} \left[\int_{\Xi_{\pm}} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \qquad (8)$$

$$\text{"}b^{*}(z,\pm)\text{"} \quad \text{for} \quad \frac{1}{H^{*}(z,\pm)} \int_{\Xi_{\mp}} \left[\int_{\Xi_{\pm}} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) \qquad (9)$$

Of course in (6) we have written:

$$\text{"}h^{*}(z,\pm)\text{"} \quad \text{for} \quad \int_{\Xi_{\pm}} N^{*}(z,\xi)\,d\Omega(\xi) \qquad (10)$$

In this regard, see (19) and (22) of Sec. 5.1, which prevent unnecessary listings of definitions associated with n-ary concepts.

Continuing with the methodical integration of the terms of (1) over Ξ_{+}, the first term on the right becomes:

$$- \int_{\Xi_{+}} \alpha(z)N^{*}(z,\xi)\,d\Omega(\xi) = - \alpha(z)h^{*}(z,+) = - \alpha(z,+)H^{*}(z,+) \qquad (11)$$

The integral term becomes:

$$\int_{\Xi_{+}} \left[\int_{\Xi} N^{*}(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi) = f^{*}(z,+)H^{*}(z,+) + b^{*}(z,-)H^{*}(z,-) \quad .$$

Finally, in like manner (and using the notation conventions agreed upon above):

$$\int_{\Xi_+}\left[\int_{\Xi_-} N^o(z,\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi')\right] d\Omega(\xi)$$

$$= f^o(z,+)H^o(z,+) + b^o(z,-)H^o(z,-) \quad .$$

Assembling these results, we have:

$$\boxed{\begin{aligned}-\frac{dH^*(z,+)}{dz} &= [f^*(z,+) - \alpha^*(z,+)]H^*(z,+) + b^*(z,-)H^*(z,-)\\ &\quad + f^o(z,+)H^o(z,+) + b^o(z,-)H^o(z,-)\end{aligned}} \tag{12}$$

In a similar manner we derive:

$$\boxed{\begin{aligned}\frac{dH^*(z,-)}{dz} &= [f^*(z,-) - \alpha^*(z,-)]H^*(z,-) + b^*(z,+)H^*(z,+)\\ &\quad + f^o(z,-)H^o(z,-) + b^o(z,+)H^o(z,+)\end{aligned}} \tag{13}$$

Analogously to (15) and (17) of Sec. 8.3, we have:

$$\alpha^*(z,\pm) = a^*(z,\pm) + s^*(z,\pm) \tag{14}$$

$$\alpha^o(z,\pm) = a^o(z,\pm) + s^o(z,\pm) \tag{15}$$

$$s^*(z,\pm) = f^*(z,\pm) + b^*(z,\pm) \tag{16}$$

$$s^o(z,\pm) = f^o(z,\pm) + b^o(z,\pm) \tag{17}$$

From the preceding four equations, we have:

$$\alpha^*(z,\pm) = a^*(z,\pm) + f^*(z,\pm) + b^*(z,\pm) \tag{18}$$

$$\alpha^o(z,\pm) = a^o(z,\pm) + f^o(z,\pm) + b^o(z,\pm) \tag{19}$$

In view of (18), equations (12) and (13) may be written alternatively as:

$$\boxed{\begin{aligned}\mp\frac{dH^*(z,\pm)}{dz} &= -[a^*(z,\pm) + b^*(z,\pm)]H^*(z,\pm) + b^*(z,\mp)H^*(z,\mp)\\ &\quad + f^o(z,\pm)H^o(z,\pm) + b^o(z,\mp)H^o(z,\mp)\end{aligned}} \quad . \tag{20}$$

The interpretation of the terms of the differential equations (20) for the diffuse component of the light field is analogous

to that for (19) of Sec. 8.3. Now, in addition to the increase of $H^*(z,\pm)$ from backscattering of the other stream, there are two additional terms representing increases: the forward and backward scattering effects of the residual irradiance flows.

To round out the system (20), we deduce from (2) of Sec. 5.2 the following pair of equations governing the residual irradiance fields:

$$\boxed{\mp \frac{dH^{o}(z,\pm)}{dz} = -\alpha^{o}(z,\pm)H^{o}(z,\pm)} \quad . \tag{21}$$

This shows that the system (20) is a pair of nonhomogeneous equations with known "source terms" as represented by the scattered residual irradiances. By (21) we can solve for $H^{o}(z,\pm)$ directly, provided $\alpha^{o}(z,\pm)$ are known. These attenuation functions are known once $\alpha(z)$ for $X(a,b)$ is given $a \le z \le b$, and the shapes of the incident radiance distributions on the boundaries of $X(a,b)$ are specified. The shape of the incident radiance distribution determines the distribution functions $D^{o}(z,\pm)$. Hence we have at once from (21):

$$H^{o}(y,-) = H^{o}(x,-) \exp\left\{ - \int_{x}^{y} \alpha^{o}(z',-)dz' \right\} \tag{22}$$

$$H^{o}(y,+) = H^{o}(z,+) \exp\left\{ - \int_{y}^{z} \alpha^{o}(z',+)dz' \right\} \tag{23}$$

for $a \le x \le y \le z \le b$. Let us write:

$$\text{"}T^{o}(x,y)\text{"} \quad \text{for} \quad \exp\left\{ - \int_{x}^{y} \alpha^{o}(z',-)dz' \right\} \tag{24}$$

and:

$$\text{"}T^{o}(y,x)\text{"} \quad \text{for} \quad \exp\left\{ - \int_{x}^{y} \alpha^{o}(z',+)dz' \right\} \tag{25}$$

whenever $a \le x \le y \le b$. $T^{o}(x,y)$ is the *transmittance factor for residual irradiance.* From these definitions we can construct the transmittance factors for diffuse irradiance. For we need only write:

$$\text{"}T^*(x,y)\text{"} \quad \text{for} \quad T(x,y) - T^{o}(x,y) \quad , \tag{26}$$

whenever $a \le x \le y \le b$. This defines the *transmittance factor for diffuse downward irradiance.* A similar definition is readily phrased for the upward diffuse irradiance. Definition (26) is the irradiance counterpart to (41) of Sec. 7.1. From (26) follows:

$$\boxed{T(x,y) = T^{o}(x,y) + T^{*}(x,y)} \tag{27}$$

which is the decomposition of the transmittance factor $T(x,y)$ into its residual and diffuse parts.

Principles of Invariance for Diffuse Irradiance

Starting with the principles of invariance (1) and (2) of Sec. 8.1 for irradiance, and using the decomposition (27) of the irradiance transmittance factors along with the irradiance decomposition (5), we can deduce the principles of invariance for the diffuse irradiance field. Thus, from (1) of Sec. 8.1:

$$\begin{aligned} H^{o}(y,+) + H^{*}(y,+) = {} & (H^{o}(z,+) + H^{*}(z,+))(T^{o}(z,y) + T^{*}(z,y)) \\ & + (H^{o}(y,-) + H^{*}(y,-))R(y,z) \end{aligned}$$

and using (23), we have:

$$\boxed{\begin{aligned} \text{I}^{*} \quad H^{*}(y,+) = {} & H^{*}(z,+)T(z,y) + H^{*}(y,-)R(y,z) \\ & + H^{o}(z,+)T^{*}(z,y) + H^{o}(y,-)R(y,z) \end{aligned}} \tag{28}$$

Similarly:

$$\boxed{\begin{aligned} \text{II}^{*} \quad H^{*}(y,-) = {} & H^{*}(x,-)T(x,y) + H^{*}(y,+)R(y,x) \\ & + H^{o}(z,-)T^{*}(x,y) + H^{o}(y,+)R(y,x) \end{aligned}} \tag{29}$$

where $a \le x \le y \le z \le b$. These are the two main principles of invariance for diffuse irradiance. The remaining two are obtained from I* and II* in the usual manner. Thus in I* let $y = a$ and $z = b$:

$$\boxed{\text{III}^{*} \quad H^{*}(a,+) = H^{o}(b,+)T^{*}(b,a) + H^{o}(a,-)R(a,b)} \tag{30}$$

and setting $x = a$, $y = b$ in II*:

$$\boxed{\text{IV}^{*} \quad H^{*}(b,-) = H^{o}(a,-)T^{*}(a,b) + H^{o}(b,+)R(b,a)} \tag{31}$$

In obtaining III* and IV* from I* and II* use was made of the convention, stated in the discussion of (30) of Sec. 7.1, that $N_{+}^{*}(b) = 0$ and $N_{-}^{*}(a) = 0$. These principles for $H^{*}(y,\pm)$ should be compared with I* through IV* of Sec. 7.1 (starting with (52) of Sec. 7.1).

Classical Models for Irradiance Fields

As illustrations of the general ideas developed in this and the preceding section we discuss some of the classical models for irradiance fields studied by early workers in the field of radiative transfer theory.

The classical equations of the two-flow irradiance field as studied by Schuster, Silberstein, Ryde, Duntley, etc., were in each case derived *de novo* for the case of the decomposed light field. By "derived *de novo*" is meant the derivation of the two irradiance equations afresh without any reference to underlying principles of radiative transfer and exactly in the intuitive manner Schuster first derived them in 1905 using physical conservation arguments on radiant flux through a very thin layer of the plane-parallel medium. The geometrical setting was the slab geometry. Furthermore, as far as assumptions on the optical properties of the slab were concerned, homogeneity and isotropy were invariably adopted. The boundaries were nonreflecting, the usual plan being that the equations were first to be solved for this case, and then an interreflection calculation, in infinite series form, was to be undertaken subsequently if desired. The diffuse light field within such models was represented by a uniform radiance distribution over Ξ_+ and Ξ_- at each point of the medium. The incident radiance distributions were invariably of two types: collimated or uniform over Ξ_- or Ξ_+.

Collimated-Diffuse Light Field Models

Our present task is to show how some of the general concepts introduced in Sec. 8.3 and Sec. 8.4 reduce to simpler forms under the applications of the assumptions customarily invoked in the classical two-flow models. For this purpose we assume that collimated radiance is incident on the upper boundary X_a of $X(a,b)$ along the direction ξ^o in Ξ_- and of magnitude N^o. Hence:

$$N^o(a,\xi) = N^o\delta(\xi - \xi^o) \tag{32}$$

for ξ^o, ξ in Ξ_-. No other sources of flux are incident on $X(a,b)$. Furthermore, we assume that the diffuse radiance distribution at each depth z in $X(a,b)$ is uniform over Ξ_+ and over Ξ_-. Hence:

$$N^*(z,\xi) = \begin{cases} f_+(z) & \text{if } \xi \text{ is in } \Xi_+ \\ \\ f_-(z) & \text{if } \xi \text{ is in } \Xi_- \end{cases} \tag{33}$$

for every z, $a \le z \le b$. The actual depth structure of the nonvanishing functions $f_\pm(z)$ is not of interest at the moment.

It follows from (37) of Sec. 7.1 that for ξ in Ξ_-:

$$N^{o}(z,\xi) = N^{o} \exp \left\{ - (z-a)\alpha / \mid \xi\cdot\mathbf{k} \mid \right\} \delta(\xi - \xi^{o}) \quad . \quad (34)$$

The distribution function $D^{o}(z,-)$ associated with $N^{o}(z,\xi)$ is given by:

$$D^{o}(z,-) = \frac{1}{\mid \xi^{o}\cdot\mathbf{k} \mid} \quad , \qquad (35)$$

which follows immediately from the definition of $D^{o}(z,-)$. In contrast to $D^{o}(z,-)$, the distribution function for the diffuse irradiance field is, by (3), (6), and (10):

$$D^{*}(z,\pm) = 2 \quad . \qquad (36)$$

To see this in more detail, observe first that:

$$\begin{aligned} h^{*}(z,\pm) &= \int_{\Xi_{\pm}} N^{*}(z,\xi)\, d\Omega(\xi) \\ &= \int_{\Xi_{\pm}} f_{\pm}(z)\, d\Omega(\xi) \\ &= f_{\pm}(z) \int_{\Xi_{\pm}} d\Omega(\xi) = 2\pi\, f_{\pm}(z) \end{aligned} \qquad (37)$$

Next, note that:

$$\begin{aligned} H^{*}(z,\pm) &= \int_{\Xi_{\pm}} N^{*}(z,\xi) \mid \xi\cdot\mathbf{k} \mid d\Omega(\xi) \\ &= f_{\pm}(z) \int_{\Xi_{\pm}} \mid \xi\cdot\mathbf{k} \mid \; d\Omega(\xi) \\ &= \pi\, f_{\pm}(z) \end{aligned} \qquad (38)$$

Then, by (6), (36) follows.

With these particular diffuse radiance distributions extant in $X(a,b)$, we can readily evaluate the attenuation, forward, and backward scattering functions by means of (7) through (9). Thus:

$$\alpha^{*}(z,\pm) = 2\alpha(z) \qquad (39)$$

$$f^*(z,\pm) = \frac{1}{\pi}\int_{\Xi_\pm}\left[\int_{\Xi_\pm}\sigma(z;\xi';\xi)\,d\Omega(\xi')\right]d\Omega(\xi) \;, \qquad (40)$$

$$b^*(z,\pm) = \frac{1}{\pi}\int_{\Xi_\mp}\left[\int_{\Xi_\pm}\sigma(z;\xi';\xi)\,d\Omega(\xi')\right]d\Omega(\xi) \;. \qquad (41)$$

The corresponding attenuating functions for the residual irradiance field associated with the collimated radiance distributions are obtained by using the residual-flux analogs to (7) through (19):

$$\alpha^o(z,-) = \alpha(z)/\,|\,\xi^o\cdot\mathbf{k}\,| \;, \qquad (42)$$

$$f^o(z,-) = \frac{1}{|\,\xi^o\cdot\mathbf{k}\,|}\int_{\Xi_-}\sigma(z;\xi^o;\xi)\,d\Omega(\xi) \;, \qquad (43)$$

$$b^o(z,-) = \frac{1}{|\,\xi^o\cdot\mathbf{k}\,|}\int_{\Xi_+}\sigma(z;\xi^o;\xi)\,d\Omega(\xi) \;. \qquad (44)$$

From the residual counterpart to (16):

$$s^o(z,-) = s(z)/\,|\,\xi^o\cdot\mathbf{k}\,| \;, \qquad (45)$$

and similarly:

$$a^o(z,-) = a(z)/\,|\,\xi^o\cdot\mathbf{k}\,| \;. \qquad (46)$$

We have retained the "z" in the notation, even though $X(a,b)$ is assumed homogeneous, to show that arbitrary depth dependence of α, σ does not destroy the otherwise simple relations holding among the attenuation functions for the residual and diffuse components of the light field. In this way we show that it is the *directional* structures of the radiance distributions which complicate the form of the associated attenuating functions, and not the depth dependence of the attenuating functions.

The early principal work on the attenuating functions was done by Ryde and Cooper in Ref. [270]. However, no account was taken of the dependence of these functions on the intricate directional structure of the radiance distributions in real media. As can be seen by an examination of (8) and (9), all dependence of f^* and b^* on N^* is wiped away by the assumption (33). Furthermore in the absence of a rigorous general definition of f and b, as given in (7) and (8) of Sec. 8.3 or (8) and (9) of this section, one was unable to deduce with rigor the various important properties of these functions, and occasionally inaccuracies arose. For example, one of Ryde's principal conclusions about the properties of f and b was that (in the present notation):

$$f^*(z,\pm) + b^*(z,\pm) = f^{\circ}(z,\pm) + b^{\circ}(z,\pm)$$

i.e., that:

$$s^*(z,\pm) = s^{\circ}(z,\pm) \quad .$$

However, it is at once clear from (14) of Sec. 8.3 that for the diffuse and residual cases:

$$s^*(z,\pm) = s(z)D^*(z,\pm) = 2s(z) \tag{47}$$

$$s^{\circ}(z,\pm) = s(z)D^{\circ}(z,\pm) = s(z)/\,|\,\xi_{\circ} \cdot \mathbf{k}\,| \tag{48}$$

so that $s^{\circ} = s^*$ if and only if $2 = 1/\,|\,\xi_{\circ} \cdot \mathbf{k}\,|$. Thus the distribution function plays an essential role in the correct study of the directional structure of radiance distributions and of the dependencies of the various attenuating functions on the radiance distributions. The connections among the functions $\alpha(z,\pm)$, $s(z,\pm)$, $a(z,\pm)$, $f(z,\pm)$, $b(z,\pm)$ and the directional structure of the light fields was not clearly understood in the early papers of the two-flow theory. The investigators were invariably preoccupied with obtaining a soluble differential equational for some particular special practical problem and sparse attention was addressed to the general logical and physical aspects of the equations.

It was not until the work of Duntley, Ref. [69], that a reasonably clear indication was obtained of the possible existence of a full family of attenuating functions that may be associated with the two-flow equations. Duntley added a new attenuating function to Ryde's list, namely (in our notation) $\alpha^{\circ}(z,-)$. Under the usual assumptions on N° and N^* it follows from (13) of Sec. 8.3 that:

$$a^{\circ}(z,-) = a(z)/\,|\,\xi^{\circ} \cdot \mathbf{k}\,| \tag{49}$$

and that:

$$a^*(z,-) = 2a(z) \quad . \tag{50}$$

Duntley concluded that $a^{\circ}(z,-)$ and $a^*(z,-)$ should differ by virtue of the difference in directional structure of N° and N^*. However, the simple connection:

$$a^*(z,-) = [2\,|\,\xi^{\circ} \cdot \mathbf{k}\,|\,]\; a^{\circ}(z,-) \tag{51}$$

that existed between these two absorption functions was not given, for lack of availability of the concept of the distribution function.

A connection between a° and a^* was noted by Hulburt in Ref. [114] for the special case where $\xi_{\circ} = -\mathbf{k}$, so that $a^*(z,-) = 2a^{\circ}(z,-)$. The preceding relation (51) and Hulburt's special observation are special cases of the general relations:

$$a^*(z,+) = [D^*(z,\pm)/D^{\circ}(z,\pm)]\; a^{\circ}(z,\pm) \tag{52}$$

$$s^*(z,\pm) = [D^*(z,\pm) / D^{\circ}(z,\pm)]s^{\circ}(z,\pm) \tag{53}$$

and these in turn are special cases of:

$$\boxed{\alpha^*(z,\pm) = [D^*(z,\pm) / D^{\circ}(z,\pm)]\alpha^{\circ}(z,\pm)} \tag{54}$$

These observations show the importance of the systematic use of the distribution function concept in the two-flow theory of irradiance fields. We shall use this concept repeatedly in subsequent discussions.

Isotropic Scattering Models

The next class of special two-flow equations to be considered is distinguished by the assumption of isotropy imposed on σ. Thus it is assumed that:

$$\sigma(z;\xi';\xi) = s/4\pi \tag{55}$$

for every ξ', ξ in Ξ. Hence all directional structure of σ is suppressed in such models. Let us examine the consequences of such an assumption. In what follows, we shall allow the directional structure of the light field to be arbitrary. We begin with the undecomposed light field. Using (55) in (7) and (8) of Sec. 8.3 we have:

$$f(z,\pm) = \frac{1}{2} D(z,\pm)s \tag{56}$$

$$b(z,\pm) = \frac{1}{2} D(z,\pm)s \tag{57}$$

Further:

$$a(z,\pm) = D(z,\pm)a \tag{58}$$

$$s(z,\pm) = D(z,\pm)s \tag{59}$$

So that

$$\alpha(z,\pm) = D(z,\pm)\alpha \tag{60}$$

From this we see that, under the assumption (55), the burden of the depth dependence of the light field is carried by the distribution functions. The associated forms of (19) of Sec. 8.3 are:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = -\frac{1}{2}[2a+s]D(z,\pm)H(z,\pm) + \frac{1}{2} sD(z,\pm)H(z,\mp)} \tag{61}$$

Since $D(z,\pm)$ clearly depends on the unknown structure of the radiance distributions in $X(a,b)$, equation (61) as it stands has unknown variable coefficients. If the usual assumption is now made that $D(z,\pm)$ are known (or that they vary in some relatively innocuous manner) then the preceding system is solvable. The original Schuster equations were of the form (61) in which the irradiances were diffuse only and such that $D^*(z,\pm) = 2$, and with source terms $h_\eta(z,\pm)$ added. The transition from (61) to its decomposed form is then attained by simply starring all quantities and adding the appropriate source terms (cf. (20)).

Connections with Diffusion Theory

It is of interest to observe that (61) is just two steps away from a steady state diffusion equation for photons. The first step toward the diffusion equation is taken by adding the members of (61), term by term. Thus the left side becomes:

$$- \frac{d}{dz} [H(z,+) - H(z,-)] = \nabla \cdot \mathbf{H}(z)$$

By virtue of (10) through (12) of Sec. 2.8 and the stratified light field condition the x and y derivatives in the divergence operation vanish. Furthermore, the sum of the first terms on the right becomes:

$$- \frac{1}{2} \left[2a + s\right] \left[D(z,+)H(z,+) + D(z,-)H(z,-)\right] = - \frac{1}{2} \left[2a + s\right] h(z)$$

The sum of the second terms on the right is:

$$\frac{1}{2} s [D(z,+)H(z,+) + D(z,-)H(z,-)] = \frac{1}{2} sh(z)$$

The first step concludes as we reduce these sums still further so that the net result is:

$$\nabla \cdot \mathbf{H}(z) = - ah(z) \quad . \tag{62}$$

The second step toward the diffusion equation is to assume that Fick's law (5) of Sec. 6.5 is valid and of the form:

$$\mathbf{H}(z) = - D\nabla h(z) \quad .$$

This assumption, as we saw in Chapter 6, holds relatively accurately in decomposed light fields. Combining this with (62) we have:

$$D \nabla^2 h(z) - a h(z) = 0 \tag{63}$$

which is the steady-state emission-free version of (7) of Sec. 6.5.

Now that we have seen the connection between (61) and the diffusion equation, we are led to wonder if (19) of Sec. 8.3 has the same property, i.e., of being connectible to

diffusion theory via the two generic steps just taken for (61). The answer is in the affirmative. However, we shall leave this matter until Sec. 8.8 in which the vectorial aspects of the irradiance field will be studied in detail.

8.5 Two-D Models for Irradiance Fields

We arrive now at the heart of the theory of irradiance fields in natural optical media, namely the two-D models in such media. The "two-D" aspect of the models refers to the radiance distributions over $\Xi_\pm$ being assigned fixed shapes so that in turn the distribution functions $D(z,\pm)$ are assigned two arbitrary fixed values: $D(\pm)$ for all z. As a result, the exact two-flow equations (19) of Sec. 8.3 and (12) of Sec. 8.4 have known depth-independent attenuation functions and so the solution procedures of those equations reduce to straightforward applications of the theory of second order ordinary differential equations with constant coefficients. The routine solution procedure of the equations can be enriched with digressions into the physical meanings of the various basic terms arising in the procedure, and we shall expend most of our efforts in the present section in such activity.

On the Depth Dependence of the Attenuating Functions

The first matter we shall take up is the depth dependence of the functions $f(z,\pm)$, $b(z,\pm)$, $\alpha(z,\pm)$, and $s(z,\pm)$, in natural optical media. The observations we shall make are designed to lay the ground work for the two-D theory. Thus our present goal is to show that the attenuation functions listed above vary relatively little with depth in homogeneous media. To begin, we consider the depth behavior of $\alpha(z,\pm)$ and $s(z,\pm)$, as this behavior is relatively simple to analyze into its physical and geometrical components. The case of $\alpha(z,\pm)$ is typical, so that we can limit our attention to it. According to (6) of Sec. 8.3 $\alpha(z,\pm)$ is the product of two factors: $\alpha(z)$ and $D(z,\pm)$. Hence the depth-variation of $\alpha(z,\pm)$ is tied to that of $\alpha(z)$ and $D(z,\pm)$. The depth variation of $\alpha(z)$ constitutes the physical component of the depth variation of $\alpha(z,\pm)$ and the distribution function $D(z,\pm)$ constitutes the geometric component of the depth variation of $\alpha(z,\pm)$. If the medium $X(a,b)$ is homogeneous, then $\alpha(z)$ is independent of z, so that any depth variation of $\alpha(z,\pm)$ is contributed by that of $D(z,\pm)$.

Now it is intuitively clear that there is generally a variation of the shape of the radiance distribution with depth z in natural waters or altitude z in the atmosphere. This variation in shape is reproduced by $D(z,\pm)$ in its own characteristic manner. It turns out that in most natural hydrosols, for example oceans and lakes, the depth variation of $D(z,\pm)$ is quite small, and what variation exists is quite regular or mild with depth. Table 1 exhibits a typical set of cases of the depth variation of $D(z,\pm)$ on clear and overcast days. These values were computed from some experimental data collected by J. E. Tyler (Ref. [298]) taken in a homogeneous part of Pend Oreille Lake, Idaho, for wavelength 480

TABLE 1

Experimentally Determined Distribution Functions

Clear Sunny Day			Completely Overcast Day		
Depth z (meters)	D(z,+)	D(z,-)	Depth z (meters)	D(z,+)	D(z,-)
4.0	2.67	1.25	3.0	2.75	1.22
10.0	2.70	1.26	12.0	2.82	1.32
16.0	2.79	1.28	24.0	2.85	1.31
28.0	2.76	1.31	36.0	2.93	1.33
40.0	2.78	1.31	49.0	2.86	1.33
53.0	2.77	1.30			

$\pm$ 64 mμ. In particular, we used the data recorded in Table 1 of Sec. 1.4, and rounded depths to integral values. The main observation we can make about the data of Table 1 is the relatively small amount of depth variation in $D(z,\pm)$ under both sunny and overcast conditions. It is also to be noted that $D(z,+)$ is approximately twice that of $D(z,-)$ and that their sum hovers in the immediate vicinity of 4. These interesting numerical relations are quite universally observed under the stated conditions and we shall study and apply such numerical regularities in Chapter 10. For the present we rest with the fact that there is an empirical basis for the two-D assumption about natural hydrosols to be made below.

It remains to make some observations on the depth behavior of the forward and backward scattering functions $f(z,\pm)$, $b(z,\pm)$. According to (7) and (8) of Sec. 8.3, the depth dependence of both of these functions is a complicated composition of a physical component contributed by $\sigma(z;\xi';\xi)$ and a geometric component associated with $N(z,\xi')$. We thus cannot separate the geometric and physical components of the depth variation of $f(z,\pm)$ and $b(z,\pm)$ as simply as in the case of $\alpha(z,\pm)$ and $s(z,\pm)$. However, we can still make a few observations that will be of help in building the two-D theory.

First of all we note that if the shape of the radiance distribution $N(z,\cdot)$ is arbitrary but fixed as z varies with depth, then $D(z,\pm)$ are independent of depth. Thus suppose:

$$N(z,\xi) = \begin{cases} f(z)g_+(\xi) & \text{if } \xi \text{ is in } \Xi_+ \\ \\ f(z)g_-(\xi) & \text{if } \xi \text{ is in } \Xi_- \end{cases} \qquad (1)$$

Then:

$$h(z,\pm) = \int_{\Xi_\pm} N(z,\xi)\,d\Omega(\xi)$$

$$= f(z) \int_{\Xi_\pm} g_\pm(\xi)\,d\Omega(\xi)$$

and:

$$H(z,\pm) = \int_{\Xi_\pm} N(z,\xi)\,|\,\xi \cdot \mathbf{k}\,|\,d\Omega(\xi)$$

$$= f(z) \int_{\Xi_\pm} g_\pm(\xi)\,|\,\xi \cdot \mathbf{k}\,|\,d\Omega(\xi) \quad .$$

Hence:

$$D(z,\pm) = \frac{\int_{\Xi_\pm} g_\pm(\xi)\,d\Omega(\xi)}{\int_{\Xi_\pm} g_\pm(\xi)\,|\,\xi \cdot \mathbf{k}\,|\,d\Omega(\xi)} \tag{2}$$

This shows that, under the fixed-shape assumption (1) on $N(z,\xi)$ the distribution functions are independent of depth.

Now what about the converse of this observation: If the distribution functions $D(z,\pm)$ are independent of depth, are the associated radiance distributions fixed in shape as a function of depth? If this were so, then Table 1 would supply the empirical evidence necessary to assert the depth independence of the shape of radiance distribution. To answer the preceding question, let us consider what constraints are imposed on $N(z,\cdot)$ when we require $D(z,-)$ to be independent of depth, say with fixed magnitude $D(-)$. All we need know about $D(-)$ at the moment is that it is not less than 1, as a perusal of its definition would show. Thus we have, directly from (5) of Sec. 8.3:

$$D(-) = \frac{\int_{\Xi_-} N(z,\xi)\,d\Omega(\xi)}{\int_{\Xi_-} N(z,\xi)\,|\,\xi \cdot \mathbf{k}\,|\,d\Omega(\xi)}$$

For $D(-)$ to be fixed is evidently a rather restrictive condition on $N(z,\cdot)$. This condition can be rewritten in the form:

$$\int_{\Xi_-} N(z,\xi)\left[D(-)\ |\ \xi \cdot \mathbf{k}\ |\ - 1\right]\ d\Omega(\xi) \tag{3}$$

This form makes it quite clear that, despite the condition imposed on it by (3), $N(z,\xi)$ can still vary in shape with depth. For suppose that we partition Ξ_- into m pieces A_i, $i = 1, \ldots, m$, over each of which $N(z,\xi)$ is essentially constant. Then (3) becomes:

$$\sum_{i=1}^{m} c_i\ N_i = 0 \tag{4}$$

where N_i is the constant value of $N(z,\cdot)$ over A_i and where we have written:

$$\text{"}c_i\text{" for } \int_{A_i} [D(-)\ |\ \xi \cdot \mathbf{k}\ |\ - 1]d\Omega(\xi)\ . \tag{5}$$

It is evident from equation (4) that there is an infinite number of ordered m-tuples $(N_1, N_2, \ldots, N_m)$ which satisfy it, even if the m-tuples are constrained to have nonnegative components, as required in the present case by the nonnegativity of N. While this may cut down on the number of multiples which satisfy (4) we certainly cannot work with the residual infinite number of possible solutions for which the constant property of $D(z,\pm)$ generally holds.

However we still have at least one trick to play in the present algebraic game with (4), one that is based on the fact that the numbers N_i in (4) are not to be drawn at random from the real number system but are to represent physical radiances typical of those found in natural waters. Therefore it is fair to impose a further condition on the N_i, other than that of nonnegativity. This additional condition is what we shall call a *monotonicity condition* and it may be stated as follows: Let "$\mathcal{N}_m$" denote the collection of all multiple solutions of (4). Then we say that $\mathcal{N}_m$ *obeys the monotonicity condition* if the members of $\mathcal{N}_m$ can be arranged in a sequence, ordered by real numbers, such that if $(N_1(z), \ldots, N_m(z))$ and $(N_1(y), \ldots, N_m(y))$ are two members of $\mathcal{N}_m$ with $y < z$, then $N_i(y) > N_i(z)$ for $i = 1, \ldots, m$. The physical origin of the monotonicity condition is clear, the real number indexing the multiples corresponds to depth in $X(a,b)$. The greater the depth z, the smaller the m radiance components of the multiple $(N_1(z), \ldots, N_m(z))$. In fact the components are to decrease monotonically with depth.

We now return to (4) armed with the monotonicity condition and require the collections $\mathcal{N}_m$ of solutions of (4) to obey this additional condition. Toward this end observe that the coefficients c_i partition into two groups: those that are positive, and those that are negative. If any of the c_i are zero, clearly a partition $\{A_1, \ldots, A_m\}$ of Ξ_- can

be re-chosen with only minor changes so the associated c_i is not zero. The sum in (4) is rearranged so as to collect together all positive terms in one group and all negative terms in another. Hence as depth z is increased the monotonicity condition requires the m-tuples in the positive group to uniformly decrease and (4) requires the sum to be zero. Hence the members of the negative group must also decrease and in such a manner as to preserve the balance of (4). A moment's reflection will show that this still leaves many solutions satisfying (4) but it is clear that the variation of the *shapes* of these distributions has been severely restricted by the imposition of the monotonicity condition. It therefore appears that, on a practical level, the depth independence of $D(z,\pm)$ entails that of the shape of $N(z,\cdot)$ in real optical media.

We shall rest the matter of the converse property of the distribution functions at this stage, having made it plausible that the observed depth independence of $D(z,\pm)$ in natural waters will imply a corresponding depth independence of the shape of the radiance distribution in natural waters because of the conditions such as nonnegativity and monotonicity imposed on the radiance distributions which are based on auxiliary physical reasons. Another condition on $N(z,\cdot)$ which may be imposed is that of *convexity* of the shape of the distribution.

The preceding discussion has shown that the problem of the converse property of the distribution function is not fully resolved and it is left to interested students of the subject to pursue. Briefly the problem is this: What conditions on $N(z,\cdot)$ in addition to (3) and the monotonicity condition must be imposed so that the *shape* of $N(z,\cdot)$ is to be depth independent? A more practical problem of comparable mathematical difficulty is: Describe the limits within which the shape of $N(z,\cdot)$ may vary when $N(z,\cdot)$ is subject to condition (3) and the monotonicity condition. Our preliminary analysis above showed that these limits may be quite narrow.

Returning now to the question of the depth independence of $f(z,\pm)$ and $b(z,\pm)$, we see that in pursuing this question we are led along essentially the same analytic and algebraic path as in the case of the distribution function just concluded, so that any solution to the converse distribution problem defined above should shed light and be directly applicable to the associated depth independence problem of $f(z,\pm)$ $b(z,\pm)$. In particular, we can reach the corresponding conclusion that a relatively small depth dependent variation in the shape of the radiance distribution can be expected if the depth variations of $f(z,\pm)$ and $b(z,\pm)$ are small, whenever we are in homogeneous natural hydrosols.

It turns out that the converse distribution problem outlined above is needed only for relatively shallow depths in homogeneous media (on the order of three or four attenuation lengths) for below such depths the asymtotic radiance distribution begins to take hold (cf. (3) of Sec. 7.10, and Chapter 10) and the distribution function becomes essentially depth independent along with $f(z,\pm)$ and $b(z,\pm)$. For if (1) holds, then in addition to (2), we can conclude that in homogeneous media:

$$f(z,\pm) = \frac{\int_{\Xi_\pm} \left[\int_{\Xi_\pm} g_\pm(\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi') \right] d\Omega(\xi)}{\int_{\Xi_\pm} g_\pm(\xi) \; | \xi \cdot \mathbf{k} | \, d\Omega(\xi)} \tag{6}$$

and

$$b(z,\pm) = \frac{\int_{\Xi_\mp} \left[\int_{\Xi_\pm} g_\pm(\xi')\sigma(z;\xi';\xi)\,d\Omega(\xi) \right] d\Omega(\xi)}{\int_{\Xi_\pm} g_\pm(\xi) \; | \xi \cdot \mathbf{k} | \, d\Omega(\xi)} \tag{7}$$

which follows from (7) and (8) of Sec. 8.3. Since the medium is homogeneous, the "z" may be dropped from "$\sigma(z;\xi';\xi)$" to emphasize the homogeneity assumption and the fact that $f(z,\pm)$ and $b(z,\pm)$ are then depth independent.

We have now arrayed enough evidence, both empirical and theoretical, to make plausible the working assumptions of the two-D theory, namely that $D(z,\pm)$ along with $f(z,\pm)$ and $b(z,\pm)$ are essentially depth independent in natural optical media over the greater part of the depth ranges of such media. Furthermore we have shown that the distribution functions are workable indices of the *shape* of radiance distributions in natural waters, so that the relatively complex changes in the shape of radiance distributions can for many practical purposes be succinctly compressed into and represented by the pair of numbers $D(z,\pm)$.

Two-D Model for Undecomposed Irradiance Fields

The two-D model for undecomposed irradiance fields takes as its foundation the general two-flow equations (19) of Sec. 8.3 and adopts the following additional assumptions. Let $X(a,b)$ be a plane-parallel medium such that:

(i) $a = 0$, $b = z_1$ with $0 \le z_1 \le \infty$

(ii) $X(0,z_1)$ is separable and its boundaries are transparent.

(iii) The radiance distributions in $X(0,z_1)$ satisfy condition (1).

(iv) $H(0,-)$ is an arbitrary irradiance, and $H(z_1,+) = 0$.

Assumption (i) merely sets the stage on a convenient slab in euclidean space with a terrestrial coordinate frame for hydrologic optics. The upper boundary is at depth 0 and the slab

may be either finite or infinite in depth. Assumption (ii) says several things at once; first of all, separability means that the ratio σ / α is independent of depth (cf. Sec. 7.12) over the range of depths $0 \leq z \leq z_1$. Therefore, by a change of the depth variable from geometric to optical depth, $X(0,z_1)$ can be rendered homogeneous in the optical depth coordinate system. We shall assume that this is done so that α and σ may be independent of depth in what follows. Other than (ii) no restrictions are made on the relative magnitudes of α and σ, or on the directional structure of σ. The required transparency of the upper and lower boundary planes X_0 and X_{z_1} eliminates the need to consider interreflection effects between these planes and the body of $X(0,z_1)$. Such an interreflection calculation is neatly dispatched by means of the interaction principle and is reserved for the latter stages of the present discussion. The assumption (iii) immediately implies, via (5) through (7), and (13) of Sec. 8.3 that all the coefficient functions in (19) of Sec. 8.3 are constants with respect to depth. We shall emphasize this consequence of (iii) throughout this discussion by writing "$a(\pm)$" and "$b(\pm)$" for the coefficients of the two-flow equations (cf. (7)) and "$D(\pm)$" for the fixed values of the distribution functions as given by (2). Finally, assumption (iv) limits the incident source flux to an arbitrary irradiance on the upper boundary of $X(0,z_1)$. The solution of (19) of Sec. 8.3 obtained under assumptions (i) through (iv) is known as the *first standard solution* of the two-D model for undecomposed irradiance fields.

Equations (19) of Sec. 8.3 reduce, under the above conditions, to:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [a(\pm) + b(\pm)]H(z,\pm) + b(\mp)H(z,\mp)} \tag{8}$$

The general solution of system (8) may be cast into the form:

$$\boxed{H(z,\pm) = m_+ g_+(\pm) e^{k_+ z} + m_- g_-(\pm) e^{k_- z}} \tag{9}$$

where $m_\pm$ are two constants which will be determined by assumption (iv), and where we have written:

$$\text{"} g_+(\pm) \text{"} \quad \text{for} \quad 1 \pm \frac{a(\mp)}{k_+} \tag{10}$$

$$\text{"} g_-(\pm) \text{"} \quad \text{for} \quad 1 \pm \frac{a(\mp)}{k_-} \tag{11}$$

and:

$$\text{"} k_\pm \text{"} \quad \text{for} \quad \frac{1}{2} \Big\{ [a(+) + b(+) - a(-) - b(-)] \pm [(a(+) + b(+) + a(-) + b(-))^2 - 4b(+)b(-)]^{1/2} \Big\} \tag{12}$$

It is easily seen from (12) that:

$$\text{if} \quad a > 0\,, \quad \text{then } k_- < 0 < k_+ \tag{13}$$

$$\text{if} \quad a = 0 \quad \text{and} \quad b(+) = b(-), \text{ then} \quad k_- = 0 = k_+ \tag{14}$$

$$\text{if} \quad a = 0 \quad \text{and} \quad b(+) > b(-), \text{ then} \quad k_- = 0 \quad \text{and} \quad k_+ = b(+) - b(-) \tag{15}$$

$$\text{if} \quad a = 0 \quad \text{and} \quad b(+) < b(-), \text{ then} \quad k_- = b(+) - b(-) \quad \text{and} \quad k_+ = 0 \tag{16}$$

In all cases, then, we have $k_- \le 0 \le k_+$. This permits us to view the term in (9) with k_+ in the exponential as a general growth term, while that with k_- may be viewed as a general decay term, when z is measured positive downward into $X(0,z_1)$. We shall henceforth assume the medium to be *nondegenerate*, i.e., $k_+ \neq k_-$, which limits our considerations to media with properties (13), (15), and (16). Degenerate media have trivial radiative transfer properties, as may be inferred from the solutions displayed below.

Putting assumption (iv) to work, we require of (9) that:

$$H(0,-) = m_+ g_+(-) + m_- g_-(-) \tag{17}$$

$$0 = m_+ g_+(+) e^{k_+ z_1} + m_- g_-(+) e^{k_- z_1} \tag{18}$$

From (17) and (18) we find that:

$$m_\pm = \mp H(0,-) g_\mp(+) e^{k_\mp z_1} \big/ \Delta(z_1) \tag{19}$$

where we have written:

$$\text{"}\Delta(z_1)\text{"} \quad \text{for} \quad g_+(+) g_-(-) e^{k_+ z_1} - g_+(-) g_-(+) e^{k_- z_1} \tag{20}$$

Therefore the *first standard solution* of the two-flow equations for $X(0,z_1)$ consists of the following two equations:

$$\boxed{H(z,\pm) = \frac{H(0,-)}{\Delta(z_1)} \left[g_+(+) g_-(\pm) e^{k_+ z_1 + k_- z} - g_-(+) g_+(\pm) e^{k_+ z + k_- z_1} \right]} \tag{21}$$

Writing the upward and downward irradiance equations separately, we have:

$$\boxed{H(z,+) = \frac{H(0,-)}{\Delta(z_1)}\, g_+(+) g_-(+) \left[e^{k_+ z_1 + k_- z} - e^{k_+ z + k_- z_1} \right]} \tag{22}$$

and

$$\boxed{H(z,-) = \frac{H(0,-)}{\Delta(z_1)}\left[g_+(+)g_-(-)e^{k_+z_1 + k_-z} - g_+(-)g_-(+)e^{k_+z + k_-z_1}\right]}$$

(23)

Observe how the boundary conditions (iv) are now built into (22) and (23). For by setting $z = z_1$ in (22), we obtain $H(z_1,+) = 0$. Setting $z = 0$ in (23) yields an identity; as expected. For all intermediate depths $0 < z < z_1$, equations (22) and (23) give the values of the irradiance fields at those depths.

It is of interest to examine the standard solutions (22) and (23) in the light of the invariant imbedding relations (7) and (8) of Sec. 8.1. Thus in (8) of Sec. 8.1, set $x = 0$, $y = z$ and $z = z_1$, so that:

$$H(z,-) = H(0,-)\mathcal{T}(0,z,z_1) \tag{24}$$

Furthermore, in (7) of Sec. 8.1 with the same substitutions:

$$H(z,+) = H(0,-)\mathcal{R}(0,z,z_1) \tag{25}$$

From (24) and (23) we have at once an explicit representation of $\mathcal{T}(0,z,z_1)$ for $X(0,z_1)$ and from (25) and (22) we have an explicit representation of $\mathcal{R}(0,z,z_1)$ for $X(0,z_1)$. Furthermore, since:

$$\mathcal{R}(0,0,z_1) = R(0,z_1)$$

it follows that:

$$\boxed{R(0,z_1) = \frac{g_+(+)g_-(+)}{\Delta(z_1)}\left[e^{k_+z_1} - e^{k_-z_1}\right]} \tag{26}$$

Similarly, since:

$$\mathcal{T}(0,z_1,z_1) = T(0,z_1)$$

it follows that:

$$\boxed{T(0,z_1) = \frac{e^{(k_+ + k_-)z_1}}{\Delta(z_1)}\left(g_+(+)g_-(-) - g_+(-)g_-(+)\right)} \tag{27}$$

We note that for infinitely deep media, i.e., for the case $z_1 = \infty$, (26) and (27) become, respectively:

$$R_\infty = \frac{(-k_-) - a(-)}{(-k_-) + a(+)} \quad , \quad T_\infty = 0$$

A question that often arises in practical applications of the principles of invariance to plane-parallel media concerns the *polarity* of the R and T factors (or R and T operators, if radiance is used) (cf. Sec. 7.12). It turns out that the two-D theory allows very detailed studies to be made of this question in the irradiance context. Thus, in the present setting the R and T factors by definition possess polarity if:

$$R(0,z_1) \neq R(z_1,0)$$

or

$$T(0,z_1) \neq T(z_1,0)$$

As we saw in Sec. 7.12 the phenomenon of polarity can arise when $X(0,z_1)$ is anisotropic or nonseparable. Since the present medium is separable, the only way polarity can arise is in the case that the inherent optical properties of $X(0,z_1)$ exhibit anisotropic structure. We have as yet made no essential use of the isotropy of $X(0,z_1)$. Suppose that we now assume the medium to be isotropic. From the polarity theorem of Sec. 7.12 we know then that the R and T *operators* do not possess polarity. But what of the *factors* $R(0,z_1)$ and $T(0,z_1)$? Since $a(\pm)$ and $b(\pm)$ play the same roles for $H(z,\pm)$ as α and σ do for $N_\pm(z)$, we should examine this question with specific reference to $a(\pm)$ and $b(\pm)$.

In preparation for the answer, let us generate the *second standard solution* by replacing (iv) above by the condition:

(v) $H(z_1,+)$ is an arbitrary irradiance, and $H(0,-) = 0$.

Therefore we are to irradiate $X(0,z_1)$ *from below*. Since the procedure for fixing $m_\pm$ is now clear, we merely state that under condition (v) with (i) through (iii) above in force, we have:

$$m_\pm = \pm H(z_1,+) g_\mp(-) / \Delta(z_1) \quad . \tag{28}$$

It follows that the *second standard solution* is:

$$\boxed{H(z,+) = \frac{H(z_1,+)}{\Delta(z_1)} \left[g_+(+) g_-(-) e^{k_+ z} - g_+(-) g_-(+) e^{k_- z} \right]} \tag{29}$$

and

$$\boxed{H(z,-) = \frac{H(z_1,+)}{\Delta(z_1)}\, g_+(-)g_-(-) \left[e^{k_+z} - e^{k_-z}\right]} \tag{30}$$

Being guided by the invariant imbedding relation once again, we arrive at:

$$\boxed{R(z_1,0) = \frac{g_+(-)g_-(-)}{\Delta(z_1)} \left[e^{k_+z_1} - e^{k_-z_1}\right]} \tag{31}$$

$$\boxed{T(z_1,0) = \frac{1}{\Delta(z_1)} \left[g_+(+)g_-(-) - g_+(-)g_-(+)\right]} \tag{32}$$

It is now of interest to compare (26) with (31) and (27) with (32). In the case of the reflectance factors we see that:

$$R(0,z_1) = R(z_1,0) \quad \text{if and only if} \quad g_+(-)g_-(-) = g_+(+)g_-(+) \tag{33}$$

and that:

$$T(0,z_1) = T(z_1,0) \quad \text{if and only if} \quad k_+ + k_- = 0 \tag{34}$$

The conditions on the right of each statement are each implied by a single statement; namely:

$$a(+) = a(-) \quad \text{and} \quad b(+) = b(-) \quad . \tag{35}$$

In this way we can find an answer to the question about polarity of the R and T factors. Toward this end, we shall agree that the medium $X(0,z_1)$ is *anistropic* with respect to the irradiance field if (35) does not hold, i.e., if its negation:

$$a(+) \neq a(-) \quad \text{or} \quad b(+) \neq b(-) \tag{36}$$

is true. Thus, if $X(0,z_1)$ is anisotropic with respect to irradiance, we are generally to expect polarity of the R and T factors, so that under standard conditions (i) through (v) a full description of the radiative transfer process in $X(0,z_1)$ by means of the irradiance field requires the four factors: $R(0,z_1)$, $R(z_1,0)$, $T(0,z_1)$, $T(z_1,0)$, as given by (26), (27), (31), and (32).

The basic equations of the two-D theory for undecomposed source-free irradiance fields have now been derived. The solutions of the special forms of the two-flow equations (8) giving rise to the two-D equations are conveniently grouped

into parts: the first and second standard solutions given, respectively by (22), (23) and (29), (30). These two groups can be assembled into one compact package by means of the invariant imbedding relation. Toward this end and in view of (24) and (25) we deduce:

$$\mathcal{R}(0,z,z_1) = \frac{g_+(+)g_-(+)}{\Delta(z_1)}\left[e^{k_+z_1+k_-z} - e^{k_+z+k_-z_1}\right] \tag{37}$$

$$\mathcal{T}(0,z,z_1) - \frac{1}{\Delta(z_1)}\left[g_+(+)g_-(-)e^{k_+z_1+k_-z} - g_+(-)g_-(+)e^{k_+z+k_-z_1}\right] \tag{38}$$

In a similar way we deduce the present forms of the complete reflectance and transmittance factors from (29) and (30). As a result we have:

$$\mathcal{R}(z_1,z,0) = \frac{g_+(-)g_-(-)}{\Delta(z_1)}\left[e^{k_+z} - e^{k_-z}\right] \tag{39}$$

$$\mathcal{T}(z_1,z,0) = \frac{1}{\Delta(z_1)}\left[g_+(+)g_-(-)e^{k_+z} - g_+(-)g_-(+)e^{k_-z}\right] \tag{40}$$

The preceding expressions can be condensed slightly by noting that the square-bracketed quantities in (38) and (40) are generalized forms of $\Delta(x)$ for suitable x. Furthermore the bracketed quantities in (37) and (39) can be represented in terms of hyperbolic functions, after suitable rearrangements. However such condensations are relatively complex and, in the final analysis, quite inessential, since the true algebraic structures of the irradiance field in plane-parallel media are those given by the invariant imbedding factors and the invariant imbedding relation to which they belong. Thus the right sides of (37) through (40) serve merely to supply the numerical magnitudes of the $\mathcal{R}$ and $\mathcal{T}$ factors. It is immaterial how simple or complex these symbolic arrangements are. In view of (7) and (8) of Sec. 8.1, the essential algebraic structures are the following:

$$H(z,+) = H(z_1,+)\mathcal{T}(z_1,z,0) + H(0,-)\mathcal{R}(0,z,z_1) \tag{41}$$

$$H(z,-) = H(0,-)\mathcal{T}(0,z,z_1) + H(z_1,+)\mathcal{R}(z_1,z,0) \tag{42}$$

or their matricial form:

$$\boxed{(H(z,+),H(z,-)) = ((H(z_1,+),H(0,-))\mathcal{M}(0,z,z_1).} \tag{43}$$

Therefore, once a tabulation of the entries of the matrix $\mathcal{M}(0,z,z_1)$ is made for a medium $X(0,z_1)$, (43) supplies the

irradiance field $H(z,\pm)$ at each depth z in $X(0,z_1)$ in terms of the incident irradiances on boundaries X_0 and X_{z_1}. A convenient means of tabulation is given by the set (3) through (6) of Sec. 8.1. One can first build up an independent tabulation of four standard R and T factors for $X(0,z_1)$ using (26), (27) and (31), (32). Then the $\mathcal{R}$ and $\mathcal{T}$ factors may be obtained in any detail as desired. The particular forms of (3) through (6) of Sec. 8.1 required in the setting of $X(0,z_1)$, are obtained by making the substitutions $x \to 0$, $y \to z$, and $z \to z_1$. However, before extensive tabulations are considered, the observations throughout Sec. 8.7 should be studied, especially those which show how the R and T factors and their invariant imbedding counterparts and generalizations, can be obtained by direct integration procedures and semigroup calculations.

Two-D Models for Internal Sources

The basic equations for the two-D theory of irradiance fields as given in (8) are written explicitly for source-free media. It is a simple matter to extend these equations to include continuously distributed internal sources in $X(0,z_1)$, and we shall now devote some attention to this matter. Before doing so, we note that the case for a finite number of discrete internal sources in $X(0,z\ )$ is readily solved using the results of Example 3 of Sec. 3.9. (See (38) of Sec. 3.9, and replace $(N_+(y), N_-(y))$ by $(H(y,+), H(y,-))$, etc.) These results, though written for the radiance functions, adapt immediately to the irradiance case. The R and T factors (26), (27), (31), and (32) and the $\mathcal{R}$ and $\mathcal{T}$ factors (37) through (4) are now used to construct the Ψ - factors by means of (20)-(23) and (31)-(34) of Sec. 3.9.

The equations (8) are adapted to the continuous internal source problem by simply adjoining the source terms $h_\eta(z,\pm)$ to the respective equations, thus:

$$\mp \frac{dH(z,\pm)}{dz} = \tau(\pm)H(z,\pm) + \rho(\mp)H(z,\mp) + h_\eta(z,\pm) \qquad (44)$$

Here we have used (11) and (12) of Sec. 8.3 to help write the present compact version of the two-D equations. The present version is the irradiance counterpart to the local forms of the principles of invariance. The connection of $h_\eta(z,\pm)$ with the source function in $X(0,z_1)$ is simply this: we have written:

$$\text{"}h_\eta(z,\pm)\text{" for } \int_{\Xi_\pm} N_\eta(z,\xi)\,d\Omega(\xi) \qquad (45)$$

and in the process of going from the equation of transfer to the two-flow equations, as explained in Sec. 8.3, $N_\eta(z,\xi)$ is converted to $h_\eta(z,\pm)$. We may note in passing that we have used the *field* radiance interpretation of N_η in (45) so as to use h_η. The alternate interpretation of N_η as a *surface* radiance would have resulted in

w_η (cf. (19) of Sec. 2.7).

To solve (44) we can make use of the solutions obtained earlier in the source-free case. The homogeneous solutions of that case will be used. In order to find the particular solutions of (44) we proceed as follows. First observe that the system (44) may be converted into two second order differential equations in $H(z,\pm)$. Thus, let us write:

$$\text{"}D\text{"} \quad \text{for} \quad \frac{d}{dz}$$

Then (44) becomes:

$$[D + \tau(+)]H(z,+) = -\rho(-)H(z,-) - h_\eta(z,+) \tag{46}$$

$$[D - \tau(-)]H(z,-) = \rho(+)H(z,+) + h_\eta(z,-) \tag{47}$$

Operating on (46) with the operator $[D - \tau(-)]$, we have:

$$\begin{aligned}[D-\tau(-)][D+\tau(+)]H(z,+) &= -\rho(-)[D-\tau(-)]H(z,-)-[D-\tau(-)]h_\eta(z,+)\\ &= -\rho(-)[\rho(+)H(z,+)+h_\eta(z,-)]\\ &\qquad - [D-\tau(-)]h_\eta(z,+)\end{aligned}$$

The second equality follows from use of (47). Simplifying this result we have:

$$\begin{aligned}[D^2+(\tau(+)-\tau(-))D+\rho(+)\rho(-)-\tau(+)\tau(-)]H(z,+) &= -[D-\tau(-)]h_\eta(z,+)\\ &\quad -\rho(-)h_\eta(z,-).\end{aligned} \tag{48}$$

Next, operating on (47) with $[D + \tau(+)]$, we have:

$$\begin{aligned}&[D^2 + (\tau(+) - \tau(-))D + \rho(+)\rho(-) - \tau(+)\tau(-)]H(z,-) =\\ &\qquad = [D + \tau(+)]h_\eta(z,-) - \rho(+)h_\eta(z,+) \quad .\end{aligned} \tag{49}$$

Let us write:

$$\text{"}Y(z,\pm)\text{"} \quad \text{for} \quad \mp[D \mp \tau(\mp)]h_\eta(z,\pm) - \rho(\mp)h_\eta(z,\mp) \tag{50}$$

and

$$\text{"}\mathcal{D}\text{"} \quad \text{for} \quad [D^2 + (\tau(+) - \tau(-))D + \rho(+)\rho(-) - \tau(+)\tau(-)] \tag{51}$$

Then (48) and (49) can be written as:

$$\boxed{\mathcal{D}H(z,\pm) = Y(z,\pm)} \tag{52}$$

The source-free, two-D equations (8) are a special case of (52). Thus (8) is equivalent to:

$$\boxed{\mathcal{D}H(z,\pm) = 0} \tag{53}$$

and is obtained by setting $h_\eta(z,\pm) = 0$. The characteristic equation associated with the differential operator $\mathcal{D}$ is:

$$k^2 + [\tau(+) - \tau(-)]k + [\rho(+)\rho(-) - \tau(+)\tau(-)] = 0 \tag{54}$$

whose solutions are:

$$k_\pm = \frac{1}{2}\left\{[\tau(-) - \tau(+)] \pm [(\tau(+) + \tau(-))^2 - 4\rho(+)\rho(-)]^{1/2}\right\} \tag{55}$$

These roots k_+, k_- are precisely those given in (12), but are now represented in terms of the local transmittance and reflectance factors $\tau(\pm)$, $\rho(\pm)$. The earlier forms can be recovered by means of (11), (12), and (18) of Sec. 8.3.

We are now ready to find the general solutions $H(z,\pm)$ of (52). First we observe that two linearly independent solutions of the homogeneous equations (53) are available in the forms:

$$e^{k_\pm z}$$

Indeed, each function satisfies (53) and the Wronskian $W(e^{k_+z}, e^{k_-z})$ of these two functions is of the form:

$$W(e^{k_+z}, e^{k_-z}) = \begin{vmatrix} e^{k_+z} & e^{k_-z} \\ k_+e^{k_+z} & k_-e^{k_-z} \end{vmatrix}$$

$$= (k_- - k_+)\exp\left\{(k_+ + k_-)z\right\}$$

for every z in the interval $[0,z_1]$. Since we have agreed to work in nondegenerate media (cf. (13)-(16) of Sec. 8.5), it follows that $k_+ \neq k_-$, and so the Wronskian does not vanish on $[0,z_1]$, thereby indicating the linear independence of the solutions.

By means of the method of variation of parameters, particular solutions of (52) are found to be of the form:

$$\frac{1}{k_+ - k_-}\int_0^z \left[e^{k_+(z-s)} - e^{k_-(z-s)}\right] Y(s,\pm)\, ds \tag{56}$$

which we will denote by "$H_p(z,\pm)$".

It follows that the general solutions of (52) are of the form:

$$H(z,\pm) = m_+ g_+(\pm) e^{k_+ z} + m_- g_-(\pm) e^{k_- z} + H_p(z,\pm) \qquad (57)$$

where $m_\pm$ are constants of integration. These constants may be determined by suitable choices of the values $H(y,\pm)$ at any two distinct levels in $X(0,z_1)$, or by knowing $H(y,\pm)$ at a given depth and $dH(y,\pm)/dy$ at the same, or generally some other depth. Still another and quite general requirement would be imposed by simultaneously specifying $H(0,-)$, $H(z_1,+)$, exactly as in the source-free case considered above (re: (19), (28)). This we shall do, our ultimate goal being a representation of $H(z,\pm)$ by means of an invariant imbedding relation, as in (43), but now taking cognizance of source terms in (44) and hence in (57). Thus, we require:

$$H(0,-) = m_+ g_+(-) + m_- g_-(-)$$

$$H(z_1,+) = m_+ g_+(+) e^{k_+ z_1} + m_- g_-(+) e^{k_- z_1} + H_p(z_1,+) \quad .$$

The requisite values of $m_\pm$ are:

$$m_\pm = \pm \frac{1}{\Delta(z_1)} \left\{ \left[H(z_1,+) - H_p(z_1,+) \right] g_\mp(-) - H(0,-) g_\mp(+) e^{k_\mp z_1} \right\} \qquad (58)$$

These values are now returned to (57) and some algebraic reductions made, keeping in mind the present goal. The results may be written as:

$$\boxed{\begin{aligned}(H(z,+),H(z,-)) = {} & (H(z_1,+),H(0,-))\mathcal{M}(0,z,z_1) \\ & - (H_p(z_1,+),0)\mathcal{M}(0,z,z_1) + (H_p(z,+),H_p(z,-))\end{aligned}} \qquad (59)$$

This is the desired set of general solutions of the equations (52). Observe that we return to the source-free case (43) if $h_\eta(z,\pm) = 0$ for every z in $[0,z_1]$. The new parts comprising the present solution are generated by the presence of sources in $X(0,z_1)$. The members of the 2×2 matrix $\mathcal{M}(0,z,z_1)$ were evaluated in (37) through (40), so that, as it stands, (59) is ready for numerical computations.

Equation (59) can be recast into an alternate form so as to achieve greater symmetry of form and also to bring out explicitly the intuitive features of the role played by the continuous sources $h_\eta(z,\pm)$ within $X(0,z_1)$. The concept of the continuous Ψ-operator, introduced in (37) of Sec. 3.9, guides the reformulation of (59) towards this new goal.

First we observe that each particular solution $H_p(z,\pm)$ may be written as a sum of two terms, using the explicit form of $Y(x,\pm)$ given in (50):

$$H_p(z,+) = -\int_0^z Q(z-s)[D-\tau(-)]h_\eta(s,+)ds - \int_0^z Q(z-s)\rho(-)h_\eta(s,-)ds \tag{60}$$

$$H_p(z,-) = \int_0^z Q(z-s)[D+\tau(+)]h_\eta(s,-)ds - \int_0^z Q(z-s)\rho(+)h_\eta(s,+)ds \tag{61}$$

Note in particular how the derivative operators $[D-\tau(-)]$ and $[D+\tau(+)]$ are to be applied to $h_\eta(s,+)$ and $h_\eta(s,-)$ in the first integrals in each of the representations (60) and (61). For brevity, we have written:

$$\text{"}Q(z-s)\text{"} \quad \text{for} \quad \frac{1}{(k_+ - k_-)}\left[e^{k_+(z-s)} - e^{k_-(z-s)}\right]. \tag{62}$$

The two equations (60) and (16) can be succinctly written in matrix form and in such a way as to make contact with the Ψ - operator concepts of example 3, Sec. 3.9. Thus:

$$(H_p(z,+),H_p(z,-)) =$$

$$= \int_0^z (h_\eta(s,+),h_\eta(s,-)) \begin{bmatrix} -[D-\tau(-)] & -\rho(+) \\ -\rho(-) & [D+\tau(+)] \end{bmatrix} Q(z-s)ds$$

Let us denote the matrix in the preceding integrand by "$\mathcal{E}(s)$". The variable s is used in the notation of $\mathcal{E}(s)$ to indicate that the derivatives are to be taken with respect to s. We agree that in the integrand the operator $\mathcal{E}(s)$ acts on $(h_\eta(s,+),h_\eta(s,-))$ (and not $Q(z-s)$), and that the result is multiplied by $Q(z-s)$. Hence:

$$(H_p(z,+),H_p(z,-)) = \int_0^z (h_\eta(s,+),h_\eta(s,-))\mathcal{E}(s)Q(z-s)ds \quad . \tag{63}$$

Now, in a similar manner we can reformulate the term $(H_p(z_1,+),0)$ in (59). Indeed, we observe first that:

$$(H_p(z_1,+),0) = (H_p(z_1,+),H_p(z_1,-))C_+$$

where C_+ is one of a pair $C_\pm$ of 2×2 matrices defined in (4) and (5) of Sec. 7.4. Now the $I_\pm$ occurring in $C_\pm$ reduce to the number 1 in the irradiance context. Hence, by the same reasoning as before:

$$(H_p(z_1,+),0) = \int_0^{z_1} (h_\eta(s,+),h_\eta(s,-))\mathcal{E}(s)C_+Q(z_1-s)\,ds \quad . \tag{64}$$

The integral in (63) can be given z_1 as an upper limit by adopting the function χ defined on the real line, where

$$\chi(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

so that:

$$(H_p(z,+),H_p(z,-)) = \int_0^{z_1} (h_\eta(x,+),h_\eta(x,-))\mathcal{E}(s)\chi(z-s)Q(z-s)ds \quad .$$

With these preliminaries established, we return to (59) and cast the particular solution terms into the form:

$$(H_p(z,+),H_p(z,-)) - (H_p(z\ ,+),0)\mathcal{M}(0,z,z\ ) =$$

$$= \int_0^{z_1} (h_\eta(s,+),h_\eta(s,-))\mathcal{E}(s)[\chi(z-s)Q(z-s)-C_+\mathcal{M}(0,z,z_1)Q(z_1-s)]\,ds$$

Let us write:

$$\text{"}\Psi(s,z)\text{"} \quad \text{for} \quad \mathcal{E}(s)[I\chi(z-s)Q(z-s)-C_+\mathcal{M}(0,z,z_1)Q(z_1-s)] \qquad (65)$$

where I is the 2×2 identity matrix, so that (59) becomes:

$$\boxed{\begin{aligned} (H(z,+)H(z,-)) &= (H(z_1,+),H(0,-))\mathcal{M}(0,z,z_1) + \\ &\quad + \int_0^{z_1} (h_\eta(s,+),h_\eta(s,-))\Psi(s,z)ds \\ 0 \leq z \leq z_1 & \end{aligned}} \qquad (66)$$

Further connections of irradiance field theory with the functional relations of invariant imbedding theory can be sought, especially with those of Sec. 7.13. Some of this research may be guided by the query: What are the specific connections between the four components of the 2×2 matrix $\Psi(s,z)$ defined in (65) and those of $\Psi(s,z\colon 0,z_1)$ studied in Sec. 7.13? Evidently the theory of Sec. 7.13 is an independent means of arriving at $\Psi(s,z)$ constructed above. By relating the two methods of approach to $\Psi(s,z)$, new understanding of invariant imbedding techniques should be forthcoming. A beginning is made in Example 10 of Sec. 8.7. Some recent developments for media with internal sources are cited in §10.

Two-D Model for Decomposed Irradiance Fields

The two-D model for *decomposed* irradiance fields, to which we now turn, was the first two-D model constructed (Ref. [221]). The prime motivation for the model was the need for a more accurate and detailed irradiance model than what was available in the classical one-D theory of irradiance fields (Sec. 8.6). The present model has the advantage of working with the diffuse irradiances generated by arbitrarily shaped radiance distributions.

It is well known that the shapes of the diffuse radiance distributions in natural media change with depth at a rate much slower than the shapes of the undecomposed distributions, thereby making the two-D assumption (1) a still more faithful reflection of the actual radiometric state of affairs in the present instance. A further advantage of the decomposed irradiance model is that the shape of the incident radiance distributions on $X(0,z_1)$ can be chosen independently of the shape of the diffuse radiance distributions throughout $X(0,z_1)$, thereby constituting a greater flexibility than the one-D model. As a result the decomposition:

$$H(z,\pm) = H^{o}(z,\pm) + H^{*}(z,\pm)$$

contains two independent terms which together are more flexible in representing actual irradiance fields. A third advantage of the two-D model for decomposed irradiance fields is that the mathematical procedures for solving the model's equations are simply special cases of the solution just studied in the case of the undecomposed irradiance field with internal sources. Indeed, the "internal sources" are now the upward and downward *primary scattered* irradiances originating from the residual irradiances $H^{o}(z,\pm)$. This may be seen by comparing (2) of Sec. 8.4 with (44). Hence replacing "$h_\eta(z,\pm)$" in (44) formally by "$f^{o}(z,\pm)H^{o}(z,\pm) + b^{o}(z,\mp)H^{o}(z,\mp)$", (66) yields the requisite general solution of the present two-D model equations.

Actually, such generality as that just noted is not usually needed in practical hydrologic optics or meteorologic optics computations. The following radiometric conditions are found to be sufficient for most purposes, and we hereby explicitly adopt them in the present study:

(vi) $N^{o}(0,\xi) = N^{o}\delta(\xi-\xi^{o})$ for ξ^{o} in Ξ_- and

$N^{o}(z_1,\xi) = 0$ for ξ in Ξ_+. Further,

$H^{*}(0,-) = 0$ and $H^{*}(z_1,+) = 0$.

The remaining conditions for the present discussion are (i) through (iii) as given for the undecomposed irradiance field equations (8). Now condition (iii) applies of course to $N^{*}(z,\cdot)$, $0 \le z \le z_1$. Therefore $X(0,z_1)$ has a collimated source of radiant flux incident on its upper boundary along the direction ξ^{o} in Ξ_-. No other sources are incident

on or in $X(0,z_1)$. As stated in (vi), the boundary conditions $H^*(0,-) = 0$ and $H^*(z\ ,+) = 0$ are also in force. In regard to related uses of these conditions, see (30) and (31) of Sec. 8.4.

With the preceding assumptions in force, (20) of Sec. 8.4 reduces to:

$$\mp \frac{dH^*(z,\pm)}{dz} = - [a^*(\pm) + b^*(\pm)]H^*(z,\pm) + b^*(\mp)H^*(z,\mp) + N^o \exp\{-\alpha z/\mu_o\}\sigma_\mp(\mu_o) \quad , \tag{67}$$

where we have written:

$$"\sigma_+(\mu_o)" \quad \text{for} \quad \mu_o f^o(z,-) \tag{68}$$

$$"\sigma_-(\mu_o)" \quad \text{for} \quad \mu_o b^o(z,-) \tag{69}$$

and in which we write:

$$"\mu_o" \quad \text{for} \quad |\xi^o \cdot \mathbf{k}| \quad . \tag{70}$$

Therefore $\sigma_\pm(\mu_o)$ are simply forward and backward scattering functions, slightly modified so as to allow the collimated radiance condition (in the form "N^o") to remain explicitly before the eye (cf. (42) and (43) of Sec. 8.4). However, in numerical work and other theoretical investigations, it is easy to see, by inspection of the various results below, how to return to the f^o and b^o concepts, when desired.

The general solution of (67) is:

$$\boxed{H^*(z,\pm) = m_+ g_+(\pm)e^{k_+ z} + m_- g_-(\pm)e^{k_- z} - N^o C(\mu_o,\pm)e^{-\alpha z/\mu_o}} \tag{71}$$

In this set of equations, the $g_\pm(\pm)$ are given in (10) and (11) with "$a^*(\pm)$" replacing "$a(\pm)$" and "$b^*(\pm)$" replacing "$b(\pm)$" everywhere (including in $k_\pm$), since we are now working with the diffuse irradiance field. For simplicity of notation, the star superscripts will be kept behind the scenes in (71). Furthermore, in (71) we have written:

$$"C(\mu_o,\pm)" \quad \text{for} \quad \frac{\sigma_\pm(\mu_o)b^*(\mp) + \sigma_\mp(\mu_o)[a^*(\mp) + b^*(\mp) \mp (\alpha/\mu_o)]}{(k_+ + \frac{\alpha}{\mu_o})(k_- + \frac{\alpha}{\mu_o})} \tag{72}$$

The quantities $C(\mu_o,\pm)$ clearly depend on the direction of incidence of the radiance distribution $N^o(0,\xi)$, ξ in Ξ_-. It is this term in (71) which gives the dependence of $H^*(z,\pm)$ on μ_o, and which permits the simulation of general incident lighting conditions. By tabulating the values $C(\mu_o,\pm)$ for

a few typical choices of α and σ, a useful set of tables of $H^*(z,\pm)$ can in turn be constructed, and these can be used via superposition calculations to simulate general incident conditions. The constants $m_\pm$ in (71) are as yet the only undetermined quantities in (71). However the remaining lighting conditions on H^* in (vi) rigidly fix the structures of m_+ and m_-. Thus it may be shown that:

$$m_\pm = \frac{N^o}{\Delta(z_1)}\left[\pm g_\mp(-)C(\mu_o,+)e^{-\alpha z_1/\mu_o} \mp g_\mp(+)C(\mu_o,-)e^{k_\mp z_1}\right] \tag{73}$$

Explicit expressions for $H^*(z,\pm)$ or for $H(z,\pm)$ are now derivable from (71) using (73). (See, e.g., (1) and (2) of Sec. 10.3.) In particular, expressions for $\mathcal{R}(0,z,z_1)$ and $\mathcal{T}^*(0,z,z_1)$ are now readily forthcoming. However, we shall not take the space here to display such representations of the complete reflectance and transmittance factors. We shall be content to exhibit, for later purposes, the standard reflectance and transmittance factors $R(0,z_1)$ and $T^*(0,z_1)$. Toward this end, by means of (30) of Sec. 8.4 and suitable substitutions, we have:

$$H^*(0,+) = H^o(0,-)R(0,z_1)$$

Since

$$H^o(0,-) = N^o\mu_o \quad ,$$

(71) and (73) combine to yield:

$$\boxed{\begin{aligned} R(0,z_1) = {} & \frac{C(\mu_o,-)g_+(+)g_-(+)}{\mu_o\Delta(z_1)}\left[e^{k_+z_1} - e^{k_-z_1}\right] \\ & + \frac{C(\mu_o,+)}{\mu_o}\left[\frac{\Delta(0)}{\Delta(z_1)}\, e^{-\alpha z_1/\mu_o} - 1\right] \end{aligned}} \tag{74}$$

Further, from (31) we have:

$$H^*(z_1,-) = H^o(0,-)T^*(0,z_1) \quad ,$$

which, with (71) and (73), combine to yield:

$$\boxed{\begin{aligned} T^*(0,z_1) = {} & \frac{C(\mu_o,+)g_+(-)g_-(-)}{\mu_o\Delta(z_1)}\left[e^{k_+z_1} - e^{k_-z_1}\right] e^{-\alpha z_1/\mu_o} \\ & + \frac{C(\mu_o,-)}{\mu_o}\left[\frac{\Delta(0)}{\Delta(z_1)}\, e^{(k_+ + k_-)z_1} - e^{-\alpha z_1/\mu_o}\right] \end{aligned}} \tag{75}$$

There is generally one other reflectance-transmittance pair of factors for upward incident flux on $X(0,z_1)$. The procedures used to find (31) and (32) may serve as a guide to find the factors for the present case.

Inclusion of Boundary Effects

We conclude this section with a description of how to include the effect of reflecting upper and lower boundaries on the medium $X(0,z_1)$. The general problem of interreflections between the body of a medium and its boundary, including the possibility of internal reflecting interfaces, was discussed in Example 6 of Sec. 3.9 and applied to the unified atmosphere-hydrosphere problem in Example 7 of that section. We shall now repeat the essential ideas of those examples, but in the irradiance context, and, so as to keep the discussion simple, we shall assume no internal interfaces.

The simplest case will be considered first: the medium $X(0,z_1)$ has only one reflecting boundary namely that at level 0, and which we shall denote by "X_0". The boundary at level z_1 will be assumed transparent. In practice, when z_1 is relatively great, the present case may be freely used even though $X(0,z_1)$ has a reflecting lower boundary. Furthermore, the only source in $X(0,z_1)$ will be the downward irradiance $H^0(0,-)$, incident on the upper boundary. We will work with the undecomposed irradiance field.

With these conditions, we have established the present interaction problem as that between two media: a plane X_0 and a slab $X(0,z_1)$. The reflectance and transmittance factors for X_0 are developed in their full generality in Sec. 3.3. In particular we use $r_\pm(x)$ and $t_\pm(x)$ as developed in (19) of Sec. 3.3. Examples of the use of these factors in the irradiance context are given in Sec. 3.4. Hence we can employ the interaction method in the present problem without further explanation.

We direct attention first to plane X_0, and enumerate all incident irradiances:

A_1: all irradiances like $H_-(0)$

A_2: all irradiances like $H_+(0)$

The set A_1 is the set of all irradiances on the upper side of X_0. The set A_2 consists of all irradiances on the underside of X_0. The set of response radiant emittances of X_0 are:

B_1: all radiant emittances like $W_-(0)$

B_2: all radiance emittances like $W_+(0)$

The four associated interaction operators for X_0 are simply the reflectance and transmittance factors $r_\pm(0)$, $t_\pm(0)$ associated with X_0:

$$s_{11} \text{ ------- } t_-(0)$$

$$s_{12} \text{ ------- } r_-(0)$$

$$s_{21} \text{ ------- } r_+(0)$$

$$s_{22} \text{ ------- } t_+(0)$$

We next direct attention to the slab $X(0,z_1)$. The class of all incident irradiances on $X(0,z_1)$ is:

A_1: all irradiances like $H(0,-)$

The class of all response radiant emittances is:

B_1: all radiant emittances like $W(0,+)$

B_2: all radiant emittances like $W(z_1,-)$

The requisite response operators s_{11} and s_{12} are in this case the numerical reflectance $R(0,z_1)$ given by (26) and $T(0,z_1)$ given by (27). According to the interaction principle applied to X_o, we have:

$$W_+(0) = H_-(0)r_-(0) + H_+(0)t_+(0) \tag{76}$$

$$W_-(0) = H_-(0)t_-(0) + H_+(0)r_+(0) \tag{77}$$

The interaction principle applied to $X(0,z_1)$ yields:

$$\left.\begin{aligned} W(0,+) &= H(0,-)R(0,z_1) \\ W(z_1,-) &= H(0,-)T(0,z_1) \end{aligned}\right\} \tag{78}$$

The auxiliary equations for the present problem are:

$$H_-(0) = H^o(0,-) \tag{79}$$

$$W_-(0) = H(0,-) \tag{80}$$

$$H_+(0) = W(0,+) \tag{81}$$

We are interested in the responses of X_o and $X(0,z_1)$ as a result of their radiometric interaction induced by $H^o(0,-)$, and so use the auxiliary equations (79) through (81) to remove as many incident quantities as possible from (76) through (78). The results are:

$$W_+(0) = H^o(0,-)r_-(0) + W(0,+)t_+(0) \tag{82}$$

$$W_-(0) = H^o(0,-)t_-(0) + W(0,+)r_+(0) \tag{83}$$

$$W(0,+) = W_-(0)R(0,z_1) \tag{84}$$

$$W(z_1,-) = W_-(0)T(0,z_1)$$

Equations (83) and (84) are autonomous, so that:

$$W_-(0) = H^o(0,-)t_-(0) + [W_-(0)R(0,z_1)]r_+(0)$$

and we have:

$$\boxed{W_-(0) = \frac{H^o(0,-)t_-(0)}{1-R(0,z_1)r_+(0)}} \tag{85}$$

From this and (84):

$$\boxed{W(0,+) = \frac{H^o(0,-)t_-(0)R(0,z_1)}{1-R(0,z_1)r_+(0)}} \tag{86}$$

From this and (82):

$$\boxed{W_+(0) = H^o(0,-)\left[r_-(0) + \frac{t_-(0)R(0,z_1)t_+(0)}{1-R(0,z_1)r_+(0)}\right]} \tag{87}$$

These results may now be used to obtain the internal irradiances $H(z,\pm)$ in $X(0,z_1)$. Indeed, by the invariant imbedding relation (43) with $H(z_1,+) = 0$ and $H(0,-) = W_-(0)$ now given by (85), we have:

$$(H(z,+),H(z,-)) = (0,H(0,-))\mathcal{M}(0,z,z_1) \quad .$$

In particular:

$$\boxed{\begin{aligned} H(z,+) &= \frac{H^o(0,-)t_-(0)}{1-R(0,z_1)r_+(0)}\mathcal{R}(0,z,z_1) \qquad (88)\\ H(z,-) &= \frac{H^o(0,-)t_-(0)}{1-R(0,z_1)r_+(0)}\mathcal{T}(0,z,z_1) \qquad (89)\end{aligned}}$$

It is interesting and instructive to pause here and view (88) and (89) in the light of the semigroup relations (52) and (53) of Sec. 3.7 which certainly apply in the irradiance context. Toward this end, observe that the factors before $\mathcal{R}$ and $\mathcal{T}$ in (88) and (89) comprise basically a complete transmittance factor. Thus suppose we write:

$$\text{"}\mathcal{T}(-1,0,z_1)\text{"} \quad \text{for} \quad \frac{t_-(0)}{1-R(0,z_1)r_+(0)}$$

and

$$"H(-1,-)" \quad \text{for} \quad H^{\circ}(0,-)$$

Then (88) is of the form:

$$H(z,+) = H(-1,-)\mathcal{T}(-1,0,z_1)\mathcal{R}(0,z,z_1) \tag{90}$$

and (89) is:

$$H(z,-) = H(-1,-)\mathcal{T}(-1,0,z_1)\mathcal{T}(0,z,z_1) \tag{91}$$

The semigroup relations show that we can write (90) and (91) as:

$$\boxed{\begin{aligned} H(z,+) &= H(-1,-)\mathcal{R}(-1,z,z_1) && (92)\\ H(z,-) &= H(-1,-)\mathcal{T}(-1,z,z_1) && (93)\end{aligned}}$$

The conceptual unity afforded by the invariant imbedding point of view is illustrated quite strikingly by this conversion of (88) and (89) into (92) and (93). As a consequence we can see that the interaction problem between the boundary X_o and $X(0,z_1)$ does not differ algebraically from that between any two subslabs $X(0,y)$, $X(y,z_1)$ of $X(0,z_1)$. In view of this, we could rework the preceding analysis so that we need not count depths in $X(0,z_1)$ starting with -1, but rather with some other fiducial depth. However, the present notation has been found convenient and workable in the discrete space setting (Ref. [251]), and shall be retained.

The reader who has followed the preceding derivation of (92) and (93) can now readily extend these results to the medium $X(0,z_1)$ which is endowed with two interacting boundaries X_o, X_{z_1}. Indeed, the invariant imbedding relation for one-parameter source-free media can be invoked at this juncture without the necessity of a fresh application of the interaction method. We merely use the semigroup relations (12) of Sec. 3.9 and the invariant imbedding relation (10) of that section applied to the one parameter medium made up of the union of X_o, $X(0,z_1)$ and X_{z_1}.

To see how such an application proceeds, let us first establish some notation. The interaction factors for X_o are as listed above. Those for X_{z_1} are $r_\pm(z_1)$, $t_\pm(z_1)$, as established generally in Sec. 3.3. Those for $X(0,z_1)$ are the four standard R and T factors found in detail above. Thus each of three interacting entities X_o, $X(0,z_1)$, and X_{z_1} is generally assigned four interaction operators: two reflectances and two transmittances. The three media together, as a class, will be denoted by "$X_3(0,z_1)$". If, further, we write "$H(-1,-)$" and "$H(z_1+1,+)$" for the downward and upward incident irradiances on the space $X_3(0,z_1)$, then the irradiance field $H(z,\pm)$ at any depth z, $0 < z < z_1$ is given by:

$$\boxed{(H(z,+),H(z,-)) = (H(z_1+1,+),H(-1,-))\mathcal{M}(-1,z,z_1+1)} \tag{94}$$

where the four components of the matrix $\mathcal{M}(-1,z,z\ +1)$ are found by decomposing them using the semigroup properties (12) of Sec. 3.9. Two examples of such decompositions have already been given in (90) and (91) for the one-boundary downward flux case. In the present two-boundary case, we have for example:

$$\mathcal{T}(-1,z,z_1+1) = \mathcal{T}(-1,0,z_1+1)\mathcal{T}(0,z,z_1+1) \tag{95}$$

The geometric setting for (95) is depicted in Fig. 8.2. Using this figure as a guide, and turning to (40) through (43) of Sec. 3.7, we see from (42) of Sec. 3.7 that:

$$\mathcal{T}(-1,0,z_1+1) = T(-1,0)[1-R(0,z_1+1)R(0,-1)]^{-1} \tag{96}$$

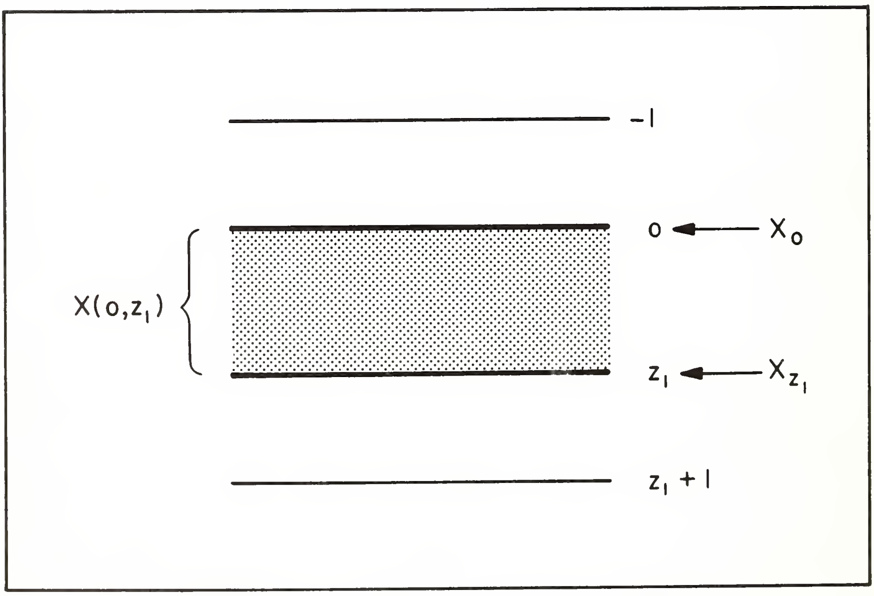


FIG. 8.2 Geometric scheme for including boundary effects of boundaries X_0 and X_{z_1}, of medium $X(0,z_1)$. Hypothetical levels labeled "-1" and "z_1 + 1" are introduced as places external to $X(0,z_1)$ from which radiant flux may irradiate $X(0,z_1)$ or at which emergent radiant flux may be measured.

Here, of course,

$$T(-1,0) = t_-(0) \tag{97}$$

$$R(0,-1) = r_+(0) \tag{98}$$

Furthermore, by (15) of Sec. 7.3, we have:

$$R(0,z_1+1) = R(0,z_1) + \mathcal{R}(0,z_1,z_1+1)T(z_1,0) \tag{99}$$

By the semigroup property of $\mathcal{R}$:

$$\mathcal{R}(0,z_1,z_1+1) = \mathcal{T}(0,z_1,z_1+1)R(z_1,z_1+1) \tag{100}$$

Here we have:

$$R(z_1,z_1+1) = r_-(z_1) \tag{101}$$

and once again by (42) of Sec. 3.7:

$$\mathcal{T}(0,z_1,z_1+1) = T(0,z_1)[1-R(z_1,0)R(z_1,z_1+1)]^{-1} \tag{102}$$

In this way we can completely and systematically analyze the factors on the right in (95) using only the main semigroup properties of $\mathcal{T}$ and $\mathcal{R}$ and the composition formulas, (40) through (43) of Sec. 3.7. The remaining three components of $\mathcal{M}(-1,z,z_1+1)$ may be analyzed similarly.

The reader interested in pursuing the radiative transfer theory of "mixed spaces"--i.e., collections of simultaneously interacting surfaces, slabs, and general media, in three and higher dimensional settings--is referred to Ref. [251], which systematically develops this discipline, known as *discrete-space radiative transfer theory*. The preceding activity is a particular instance of application of discrete-space theory. We shall return to this matter in Example 5 of Sec. 8.7 and systematize the preceding boundary effects procedure for irradiance settings.

8.6 One-D and Many-D Models

We have reached a stage in the development of the two-flow models for irradiance fields where we have enough machinery to readily examine two extreme tvpes of irradiance models used in practice and theory. The first of these types is the *one-D* model which is the modern counterpart to the early Schuster-type equations associated with light fields having spherical radiance distributions. The other extreme is the *many-D* model for irradiance fields which pushes the two-flow irradiance theory to its logical limit and can serve as the bridge back to the domain of ideas developed in Chapter 7. The explicit formulations of each of these extreme types is of importance to the subsequent developments of this work and therefore will be studied in some detail.

One-D Models for Undecomposed Irradiance Fields

The one-D model for undecomposed irradiance fields takes as its foundation (8) of Sec. 8.5 and adopts the following assumptions: Let $X(a,b)$ be a plane-parallel medium such that:

(i) $a = 0, \quad b = z_1$ with $0 \le z_1 \le \infty$

(ii) $X(0,z_1)$ is separable and its boundaries are transparent.

(iii) The radiance distributions in $X(0,z_1)$ satisfy condition (1) of Sec. 8.5 with $c\,g_+(\xi) = g_-(\xi)$, for some positive constant c.

(iv) $H(0,-)$ is an arbitrary irradiance, and $H(z_1,+) = 0$.

A comparison of these assumptions with the corresponding set (i) through (iv) in Sec. 8.5 just before (8) of that section shows that the main change is in the uniformization of the radiance distribution: It is now to have the same shape over Ξ_+ and Ξ_-. The shape may be spherical, elliptical, or any arbitrary shape imaginable; however, the crucial point is that the shapes are the same in the upper and lower hemispheres of Ξ.

The immediate consequences of assumption (iii) are the following. First we have:

$$D(+) = D(-) \quad , \tag{1}$$

which comes from (2) of Sec. 8.5. Suppose we write "D" for this common value of $D(\pm)$. As a result, (6), (13), and (14) of Sec. 8.3 imply:

$$\alpha(+) = \alpha(-) = \alpha D \tag{2}$$

$$a(+) = a(-) = aD \tag{3}$$

$$s(+) = s(-) = sD \tag{4}$$

Finally, (6) and (7) of Sec. 8.5 show that:

$$f(+) = f(-) \tag{5}$$

$$b(+) = b(-) \tag{6}$$

We shall write "f" and "b" for these common values of $f(\pm)$, $b(\pm)$. These observations make it clear that the general effect of (iii) is to induce a systematic collapse of complexity throughout Sec. 8.5. Some of the resultant condensations will now be surveyed.

The equations for the irradiance field (8) of Sec. 8.5 now take the form:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [aD + b]H(z,\pm) + bH(z,\mp)} \tag{7}$$

The first discernible effect of the one-D assumption on the solution of these equations is in the structure of k_+ and k_-, as given in (12) of Sec. 8.5. We now have:

$$k_{\pm} = \pm [(aD + b)^2 - b^2]^{1/2}$$

$$= \pm [aD(aD + 2b)]^{1/2} \quad . \tag{8}$$

If the radiance field is assumed spherical (as it occurs very nearly deep inside media with high s/α ratio) then $D = 2$ and:

$$k_{\pm} = \pm 2[a(a + b)]^{1/2} \quad .$$

If the radiance field is assumed nearly collimated vertically (as it occurs very nearly deep inside media with low s/α ratio) then $D = 1$ and:

$$k_{\pm} = \pm [a(a + 2b)]^{1/2} \tag{9}$$

Observe that for the one-D theory, having $a = 0$ requires $k_{\pm} = 0$. However, in the two-D theory, matters need not be so simple (cf. (13) through (16) of Sec. 8.5). For brevity we shall henceforth write:

$$\text{"k" for } k_+$$

so that:

$$k_- = - k \tag{10}$$

Observe that in the one-D theory, whenever $b \leq f$, we have from (8):

$$k \leq \alpha D \tag{11}$$

which represents a fundamental inequality throughout radiative transfer theory (cf. (7) of Sec. 6.6 and (26) of Sec. 9.2). We next find that $g_{\pm}(\pm)$ in (10) and (11) of Sec. 8.5 take the forms:

$$g_+(\pm) = 1 \pm \frac{aD}{k} \tag{12}$$

$$g_-(\pm) = 1 \mp \frac{aD}{k} \tag{13}$$

In other words:

$$g_+(+) = g_-(-) = 1 + \frac{aD}{k} \quad , \tag{14}$$

the common value for which we shall write "g_+"; and we shall write "g_-" for the common value of:

$$g_+(-) = g_-(+) = 1 - \frac{aD}{k} \tag{15}$$

From this and (9) of Sec. 8.5, we see that the solutions of (7) can be written:

$$\boxed{H(z,\pm) = m_+ g_\pm e^{kz} + m_- g_\mp e^{-kz}} \tag{16}$$

From this in turn we see that the system determinant $\Delta(z_1)$ in (20) of Sec. 8.5 becomes:

$$\Delta(z_1) = \left(1 + \frac{aD}{k}\right)^2 e^{kz_1} - \left(1 - \frac{aD}{k}\right)^2 e^{-kz_1} \tag{17}$$

so that:

$$\Delta(0) = \frac{4aD}{k} \quad .$$

The irradiance fields, as given by (22) and (23) of Sec. 8.5, now take the forms:

$$H(z,+) = \frac{H(0,-)}{\Delta(z_1)} \left(1 - \frac{a^2D^2}{k^2}\right) \left[e^{k(z_1-z)} - e^{-k(z_1-z)}\right] \tag{18}$$

$$H(z,-) = \frac{H(0,-)}{\Delta(z_1)} \Delta(z_1 - z) \tag{19}$$

An important constant in the one-D theory is the reflectance of an infinitely deep medium $X(0,\infty)$. This constant may be obtained from (26) of Sec. 8.5 by going to the limit as $z_1 \to \infty$. Thus if we write:

$$\text{"}R_\infty\text{"} \quad \text{for} \quad \lim_{z_1 \to \infty} R(0,z_1) \tag{20}$$

and observe that:

$$\lim_{z_1 \to \infty} \frac{e^{kz_1}}{\Delta(z_1)} = \frac{1}{g_+^2} \quad ,$$

then (26) of Sec. 8.5 implies:

$$R_\infty = \frac{g_+ g_-}{g_+ g_+} = \frac{g_-}{g_+} = \frac{1 + \frac{a(-)}{k_-}}{1 - \frac{a(+)}{k_-}} \quad .$$

Alternatively, in view of (14) and (15) this may be written:

$$R_\infty = \frac{1 - \frac{aD}{k}}{1 + \frac{aD}{k}} = \frac{k-aD}{k+aD} \tag{21}$$

Another important feature of the one-D model for $X(0,\infty)$ consists in its representations of $H(z,\pm)$. To obtain these representations, it is sufficient to observe that:

$$\lim_{z_1 \to \infty} \frac{\Delta(z_1 - z)}{\Delta(z_1)} = \lim_{z_1 \to \infty} \frac{g_+^2 \, e^{k(z_1 - z)}}{\Delta(z_1)} = e^{-kz}$$

Then (19) supplies the requisite representation for $H(z,-)$ in $X(0,\infty)$:

$$\boxed{H(z,-) = H(0,-)e^{-kz}} \quad , \tag{22}$$

and from (18) and (22);

$$\boxed{H(z,+) = H(z,-)R_\infty = H(0,+)e^{-kz}} \quad , \tag{23}$$

which holds in $X(0,\infty)$.

One-D Model for Internal Sources

The reductions from the two-D to the one-D model in the case of internal sources proceeds generally as the undecomposed light field discussion just completed. Of course, the new reduction features should be the one-D forms of the particular source solutions (56) of Sec. 8.5. To illustrate these reductions, suppose that there is a set of sources uniformly distributed throughout $X(0,z_1)$, so that $h_\eta(z,\pm)$ is independent of depth and that $h_\eta(z,+) = h_\eta(z,-)$. Let "$h_\eta$" denote the constant value. Then (50) of Sec. 8.5 becomes:

$$Y(s,\pm) = (\tau - \rho)h_\eta \quad , \tag{24}$$

so that (56) of Sec. 8.5 reduces to:

$$H_p(z,\pm) = -\frac{h_\eta(\tau - \rho)}{2k} \int_0^z \left[e^{k(z-s)} - e^{-k(z-s)}\right] ds \tag{25}$$

One-D Model for Decomposed Irradiances

The reduction of the two-D equations for decomposed irradiance fields proceeds similarly to the undecomposed case, starting with the same conditions (i) through (iii) now applied to N^*. In particular, the attenuation coefficients become:

$$\alpha^*(+) = \alpha^*(-) = \alpha D^* \tag{26}$$

$$a^*(+) = a^*(-) = aD^* \tag{27}$$

$$s^*(+) = s^*(-) = sD^* \tag{28}$$

where "D^*" denotes the common value:

$$D^*(+) = D^*(-) \tag{29}$$

and

$$f^*(+) = f^*(-) \tag{30}$$

$$b^*(+) = b^*(-) \tag{31}$$

Furthermore:

$$k_\pm = \pm [aD^*(aD^* + 2b^*)]^{1/2} \tag{32}$$

These equalities show that the decomposed case develops in a manner that is exactly parallel to the undecomposed case, as far as the homogeneous part of the solution goes. The particular part of the solution, as embodied in $C(\mu_o,\pm)$ (cf. (72) of Sec. 8.5), is now such that:

$$C(\mu_o,\pm) = \frac{\sigma_\pm(\mu_o)b^* + \sigma_\mp(\mu_o)\left[a^* + b^* \mp (\alpha/\mu_o)\right]}{\left(\dfrac{\alpha^2}{\mu_o^2} - k^2\right)} \tag{33}$$

Finally, we note that:

$$g_+(+) = g_-(-) = 1 + \frac{aD^*}{k} \quad , \tag{34}$$

the common value of which may be denoted by "g_+". Further, we denote by "g_-" the common value of:

$$g_+(-) = g_-(+) = 1 - \frac{aD^*}{k} \tag{35}$$

Equation (20) of Sec. 8.4 then reduces to:

$$\boxed{\mp \frac{dH^*(z,\pm)}{dz} = -[a^* + b^*]H^*(z,\pm) + b^*H^*(z,\mp) + f^oH^o(z,\pm) + b^oH^o(z,\mp)} \tag{36}$$

The general solution of the homogeneous part of (36) is identical in form to (16). The case of $X(0,\infty)$ is also of interest for decomposed irradiance fields. Suppose we denote by "$R_\infty(\mu_o)$" the present counterpart to R_∞ in (20). Then, parallel to (21), we have:

$$\begin{aligned} R_\infty(\mu_o) &= \frac{C(\mu_o,-)}{\mu_o}\left[\frac{g_-}{g_+} - \frac{C(\mu_o,+)}{C(\mu_o,-)}\right] \\ &= \frac{C(\mu_o,-)}{\mu_o}\left[R_\infty - \frac{C(\mu_o,+)}{C(\mu_o,-)}\right] \end{aligned}$$

where R_∞ now uses starred distribution coefficients. The respresentation of the irradiance field in $X(0,\infty)$ is readily obtained by using (73) of Sec. 8.5 to observe that:

$$\lim_{z_1\to\infty} m_+ = 0$$

and:

$$\lim_{z_1\to\infty} m_- = N^o \frac{C(\mu_o,-)}{g_+} ,$$

so that (71) reduces to:

$$H^*(z,+) = N^o\left[R_\infty C(\mu_o,-)e^{-kz} - C(\mu_o,+)e^{-\alpha z/\mu_o}\right] \tag{37}$$

$$H^*(z,-) = N^o\, C(\mu_o,-)\left[e^{-kz} - e^{-\alpha z/\mu_o}\right] \tag{38}$$

These equations show that eventually the ratio $H^*(z,+)/H^*(z,-)$ approaches R_∞. This ratio settles to R_∞ as soon as the effects of the collimated light, due to $N^o(0,\xi)$, have died away. It is noteworthy that the structures of (37) and (38) are identical to their counterparts in the full two-D theory (cf. also (1) and (2) of Sec. 10.3).

Many-D Models

There are generally two ways in which to gain insight into a physical theory: to work out numerical examples or special cases of the theory, and second, to generalize the theory to see it from a broader perspective. In this and the preceding section we have gained insight into the two-flow

theory by pursuing its special ramifications in the various two-D and one-D models. We close this section with an activity of the second kind, that is, by finding the immediate generalization of the two-D theory to the so-called "many-D" theory.

The point of departure for the two-flow equations was the steady-state source-free version of the transfer equation (1) of Sec. 8.3. We may use this equation once again as the starting point for the present discussions. However, very little extra effort will be expended if instead we start with the time-dependent transfer equation with sources, and derive the many-D theory from that. This we shall do. Let X be an arbitrary optical medium. First we write down:

$$\frac{1}{v}\,\frac{N(x,\xi,t)}{\partial t} + \xi\cdot\nabla N(x,\xi,t) =$$

$$= -\,\alpha(x,t)N(x,\xi,t) + \int_{\Xi} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi') + N_\eta(x,\xi,t). \tag{39}$$

Next, we partition Ξ into n disjoint subsets, $\Xi_1, \ldots, \Xi_n$, $n \geq 2$. For example if $n = 2$ and $\Xi_1 = \Xi_+$ and $\Xi_2 = \Xi_-$, then we would return to the usual two-flow setting. If $n = 2$ and $\Xi_1 = \{\xi : \xi \cdot \mathbf{n} > 0\} = \Xi(\mathbf{n})$, and $\Xi_2 = \{\xi : \xi \cdot \mathbf{n} < 0\} = \Xi(-\mathbf{n})$, then we would have a two-flow setting based on the partition around $\mathbf{n}$ instead of around the unit vector $\mathbf{k}$ along the z-axis. (See Fig. 8.3(a).) Thus, assuming a general partition $\{\Xi_1, \Xi_2, \ldots, \Xi_n\}$ of Ξ with respect to a unit vector $\mathbf{n}$, as in Fig. 8.3(b), we proceed to generalize (5) through (8) of Sec. 8.3 and related concepts to these. First we write: for each j, $1 \leq j \leq n$:

$$\text{"}h_j(x,t)\text{" for } \int_{\Xi_j} N(x,\xi,t)\,d\Omega(\xi) \tag{40}$$

and

$$\text{"}\mathbf{H}_j(x,t)\text{" for } \int_{\Xi_j} N(x,\xi,t)\,\xi\,d\Omega(\xi) \tag{41}$$

The vector $\mathbf{H}_j(x,t)$ is the irradiance vector induced at x at time t by the radiance distribution restricted to Ξ_j. The net irradiance induced by this vectorial flux on a surface with inner normal $\mathbf{n}$ at x and time t is:

$$\mathbf{n} \cdot \mathbf{H}_j(x,t) \tag{42}$$

and which we shall designate by "$H_j(x,\mathbf{n},t)$". Next we write

$$\text{"}D_j(x,\mathbf{n},t)\text{" for } \frac{h_j(x,t)}{H_j(x,\mathbf{n},t)} \tag{43}$$

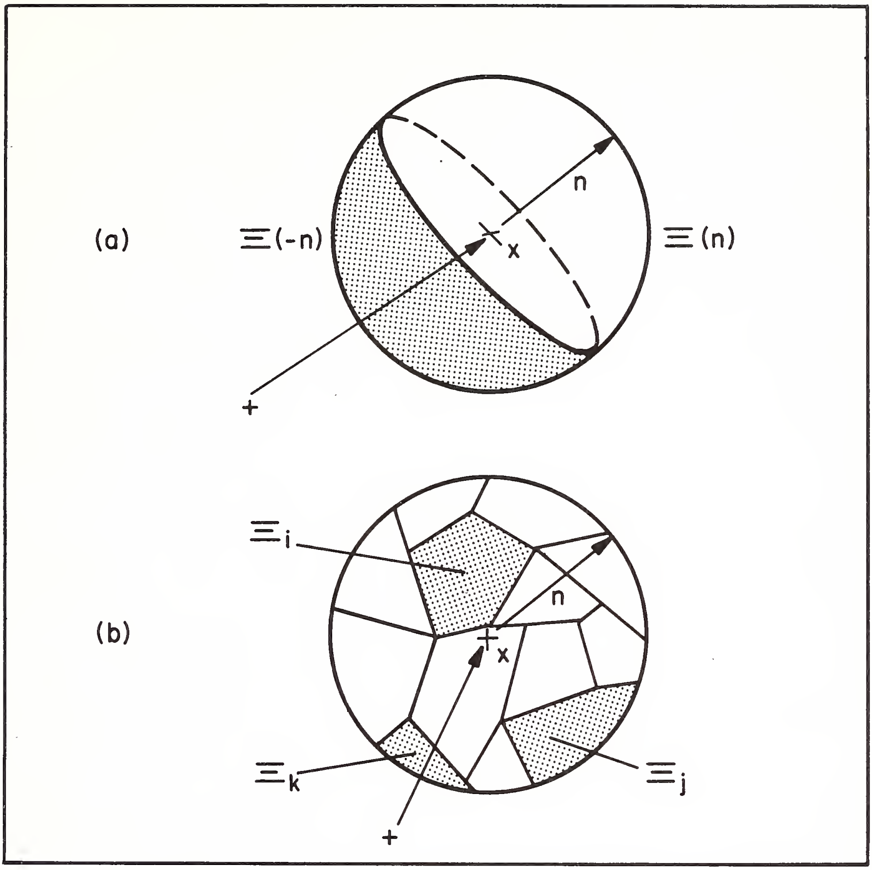


FIG. 8.3 Illustrating a two-flow decomposition of direction space Ξ, as in part (a); and a many-flow composition as in part (b). Each gives rise to a set of irradiance equations at point x in an optical medium.

The attenuating functions $\alpha_j(x,t)$, $s_j(x,t)$, $a_j(x,t)$, $j = 1$, ..., n are defined analogously to (6), (13), and (14) of Sec. 8.3. Thus, e.g., we now write:

$$\text{"}\alpha_j(x,\mathbf{n},t)\text{"} \quad \text{for} \quad \alpha(x,t)D_j(x,\mathbf{n},t) \quad . \tag{44}$$

The four forward and backward scattering forms of the two-flow theory sublimate into the following n^2 quantities:

$$\frac{1}{H_k(x,\mathbf{n},t)} \int_{\Xi_j} \left[\int_{\Xi_k} N(x,\xi',t)\sigma(x;\xi';\xi,t)\,d\Omega(\xi') \right] d\Omega(\xi) \tag{45}$$

which we shall denote by "$s_{jk}(x,\mathbf{n},t)$" for $j = 1, \ldots, n$; $k = 1, \ldots, n$. Finally, integrating (39) over Ξ_j, we find:

$$\boxed{\frac{1}{v}\frac{\partial}{\partial t}\left[D_j H_j\right] + \nabla \cdot \mathbf{H}_j = -\alpha_j H_j + \sum_{k=1}^{n} s_{jk} H_k + h_{\eta,j} \qquad j = 1, \ldots, n} \tag{46}$$

where, for brevity, space and time coordinates have been suppressed, and where in general we have written:

$$\text{"}h_{\eta,j}(x,\mathbf{n},t)\text{" for } \int_{\Xi_j} N_\eta(x,\xi,t)\,d\Omega(\xi) \tag{47}$$

The system (46) is most useful when there exists a one parameter family of space-filling surfaces (in order words, a one-parameter optical medium) over which the $\mathbf{H}_j$ and optical properties are constant valued. For example, spherical and cylindrical media are used occasionally in practice, and to which (46) may be applied. Then a coordinate system can usually be established in the medium, as in the plane-parallel case, so that the divergence term reduces to a single derivative of a suitable component of $\mathbf{H}_j$. For example, the reader will find it instructive to obtain $\nabla \cdot \mathbf{H}$ in spherical and cylindrical coordinate systems, and illustrate (46) for a two-flow setting in these two coordinate systems.

The general system can be cast, as it stands, into a form more nearly resembling the homogeneous terms of (19) of Sec. 8.3. To see this, write:

$$\begin{aligned} &\text{"}f_j\text{" for } s_{jj} \\ &\text{"}b_{jk}\text{" for } s_{jk} \end{aligned} \tag{48}$$

when $j \neq k$; also write:

$$\text{"}b_j\text{" for } \sum_{k \neq j}^{n} s_{kj} \tag{49}$$

where "$\sum_{k\neq j}^{n}$" stands for the sum over all n indices k except j.

Furthermore, write:

$$\text{"}\alpha_j\text{" for } \alpha D_j \tag{50}$$

$$\text{"}a_j\text{" for } a D_j \tag{51}$$

$$\text{"}s_j\text{" for } s D_j \tag{52}$$

Then it follows that:

$$f_j + b_j = s_j \quad , \tag{53}$$

$$\alpha_j = a_j + s_j = a_j + f_j + b_j \quad , \tag{54}$$

and (46) can then be cast into the form:

$$\boxed{\begin{gathered} \frac{1}{v}\frac{\partial}{\partial t}\left[D_j H_j\right] + \nabla \cdot \mathbf{H}_j = -\left[a_j + b_j\right] H_j + \sum_{k \neq j}^{n} b_{jk} H_k + h_{\eta,j} \\ j = 1 \ , \ \ldots \ , \ n \ . \end{gathered}} \tag{55}$$

which establishes the final generalization. By letting $n \to \infty$ such that $\max \{ \Omega(\Xi_j) : j = 1 \ , \ \ldots \ , \ n \} \to 0$, (55) returns to the equation of transfer (39), and the circle is complete.

8.7 Invariant Imbedding Concepts for Irradiance Fields

The formulations of the preceding sections of this chapter can be placed into deeper perspective when viewed from the general standpoint of the invariant imbedding concepts developed in Chapter 7. In this section we select some of the results obtained in this chapter to be so viewed. The main purpose of this activity is to indicate the conceptual and the numerical advantages gained by adopting the invariant imbedding point of view: Unsuspected symmetries and connections between the solutions of the two-flow equations spring into view when led in their general directions by the algebraic equations stored up in Sec. 7.4; furthermore the existence of quite general differential equations collected in Sec. 7.5 are now seen to hold also among the components of the irradiance field, and of the R and T factors. Furthermore, the semigroup and group-theoretic methods of Sec. 7.8 through Sec. 7.10 are awaiting their systematic translation into the irradiance context. In general, according to our observations in Sec. 8.2, *all the functional equations derived in Chapter 7 from the local or global forms of the principles of invariance also hold for the irradiance context.* Therefore, to be thorough, we could, in principle, repeat virtually all of Chapter 7 in the present section. But this is a lavish and unnecessary task for the purposes of the present work. By leaving it undone, we allow room for the few pertinent remarks made below and, more importantly, encourage students of the subject to explore such matters on their own and perhaps find new and interesting facets to develop and use. The selected examples below will indicate a few of the possible modes of approach.

Example 1: $\mathcal{R}$ and $\mathcal{T}$ Factors in Two-D Models

Equations (24) through (43) of Sec. 8.5 represent the results of a tentative, initial excursion into the invariant imbedding domain of the reflectance and transmittance concepts associated with the two-D theory. We now develop the full structure of these $\mathcal{R}$ and $\mathcal{T}$ factors, being guided by the basic equations of the invariant imbedding relation in the irradiance case, namely (7) and (8) of Sec. 8.1. Our main goal is to represent the four $\mathcal{R}$ and $\mathcal{T}$ factors for an arbitrary subslab $X(x,z)$ of $X(a,b)$ in terms of the local scattering and absorbing properties of $X(a,b)$.

We begin with the general solution (9) of Sec. 8.5 for the irradiance field at level y in $X(a,b)$. Equations (7) and (8) of Sec. 8.1 state that the irradiances $H(y,\pm)$ are a linear combination of the upward irradiance at level z and the downward irradiance at level x, the coefficients of the combination being the complete reflectance ($\mathcal{R}$) and transmittance ($\mathcal{T}$) factors for $X(x,z)$. Evidently, m_+ and m_- in (9) of Sec. 8.5 hold the key to determining $\mathcal{R}(x,y,z)$, $\mathcal{T}(x,y,z)$, $\mathcal{R}(z,y,x)$, and $\mathcal{T}(z,y,x)$. Therefore, since (9) of Sec. 8.5 holds for all depths z in $X(a,b)$ we have:

$$H(x,-) = m_+g_+(-)e^{k_+x} + m_-g_-(-)e^{k_-x} \tag{1}$$

$$H(z,+) = m_+g_+(+)e^{k_+z} + m_-g_-(+)e^{k_-z} \tag{2}$$

These two equations can be solved for m_+, m_-. The results are:

$$m_\pm = \pm\left[H(z,+)g_\mp(-)e^{k_\mp x} - H(x,-)g_\mp(+)e^{k_\mp z}\right]\Big/\Delta(x,z) \tag{3}$$

where we have written:

$$\text{"}\Delta(x,z)\text{" for } g_+(+)g_-(-)e^{k_+z + k_-x} - g_+(-)g_-(+)e^{k_+x + k_-z} \tag{4}$$

These solutions are the full symmetric forms of (19) and (28) of Sec. 8.5. The earlier forms are obtained by setting to zero the appropriate irradiances in (3).

Now, from (9) of Sec. 8.5, we can write:

$$H(y,\pm) = m_+g_+(\pm)e^{k_+y} + m_-g_-(\pm)e^{k_-y} \tag{5}$$

with $m_\pm$ as given in (3). Therefore, after some algebraic reductions to the forms (7) and (8) of Sec. 8.1, we find that for $a \le x \le y \le z \le b$:

$$\mathcal{R}(x,y,z) = \frac{g_+(+)g_-(+)}{\Delta(x,z)} \left[e^{k_+z + k_-y} - e^{k_+y + k_-z}\right] \tag{6}$$

$$\mathcal{R}(z,y,x) = \frac{g_+(-)g_-(-)}{\Delta(x,z)} \left[e^{k_+y + k_-x} - e^{k_+x + k_-y}\right] \tag{7}$$

$$\mathcal{T}(x,y,z) = \frac{\Delta(y,z)}{\Delta(x,z)} \tag{8}$$

$$\mathcal{T}(z,y,x) = \frac{\Delta(x,y)}{\Delta(x,z)} \tag{9}$$

These are the desired representations of the $\mathcal{R}$ and $\mathcal{T}$ factors for $X(x,z)$ in $X(a,b)$. It will be instructive for the reader to verify that the basic semigroup relations (52) and (53) of Sec. 3.7 hold in the present irradiance context, using (6) through (9). From these representations we can at once derive the standard R and T factors for $X(x,z)$. Thus, being guided by (9) through (12) of Sec. 8.1:

$$\mathcal{R}(x,x,z) = R(x,z) = \frac{g_+(+)g_-(+)}{\Delta(x,z)} \left[e^{k_+z + k_-x} - e^{k_+x + k_-z}\right] \tag{10}$$

$$\mathcal{R}(z,z,x) = R(z,x) = \frac{g_+(-)g_-(-)}{\Delta(x,z)} \left[e^{k_+z + k_-x} - e^{k_+x + k_-z}\right] \tag{11}$$

$$\mathcal{T}(x,z,z) = T(x,z) = \frac{\Delta(z,z)}{\Delta(x,z)} \tag{12}$$

$$\mathcal{T}(z,x,x) = T(z,x) = \frac{\Delta(x,x)}{\Delta(x,z)} \tag{13}$$

Furthermore from (6) through (9) we see that:

$$\mathcal{R}(x,z,z) = 0 \qquad \mathcal{R}(z,x,x) = 0 \tag{14}$$

$$\mathcal{T}(x,x,z) = 1 \qquad \mathcal{T}(z,z,x) = 1 \tag{15}$$

We can simplify (10) through (13) somewhat by making use of the homogeneity of $X(a,b)$. On the grounds of homogeneity we would expect the four $\mathcal{R}$ and $\mathcal{T}$ factors to be invariant under a vertical displacement of an amount d, as long as the vertically displaced slab remains within $X(a,b)$. To prove this rigorously, we need only first observe that:

$$\Delta(x+d,z+d) = \Delta(x,z)e^{(k_+ + k_-)d} \tag{16}$$

It follows at once from this and (6) through (9) that:

$$\mathcal{R}(x+d,y+d,z+d) = \mathcal{R}(x,y,z) \tag{17}$$

$$\mathcal{T}(x+d,y+d,z+d) = \mathcal{T}(x,y,z) \tag{18}$$

and similarly for $\mathcal{R}(z,y,x)$ and $\mathcal{T}(z,y,x)$. As a result, the standard R and T factors inherit this translation invariance:

$$R(x+d,z+d) = R(x,z) \tag{19}$$

$$T(x+d,z+d) = T(x,z) \tag{20}$$

and similarly for $R(z,x)$ and $T(z,x)$. However it is clear that the R and T factors generally possess polarity in the two-D theory, so that while $R(x,z) = R(0,z-x)$, $R(z,x) = R(z-x,0)$, we still need not have $R(0,z-x) = R(z-x,0)$, as an inspection of (10) through (13) will show.

In view of (16) we have:

$$\Delta(x,z) = \Delta(x-x,z-x) = \Delta(0,z-x)e^{+(k_+ + k_-)x} = \Delta(z-x)e^{+(k_+ + k_-)x} \tag{21}$$

In other words, we can strictly do without $\Delta(x,z)$ and retain only $\Delta(z)$, as defined in (20) of Sec. 8.5. Thus, the use of one or the other type of delta is optional.

Example 2: $\mathcal{R}$ and $\mathcal{T}$ Factors in One-D Models

By allowing the one-D assumptions (Sec. 8.6) to enter into the terms (3) through (21), some interesting and useful symmetries arise. The key symmetry is:

$$k_+ + k_- = 0 \tag{22}$$

In view of (21) and (22) we have also:

$$\Delta(x,z) = \Delta(0,z-x) = \Delta(z-x) \tag{23}$$

It follows that:

$$R(0,z-x) = R(x,z) = R(z,x) = R(z-x,0) \tag{24}$$

$$T(0,z-x) = T(x,z) = T(z,x) = T(z-x,0) \tag{25}$$

The complete $\mathcal{R}$ and $\mathcal{T}$ factors then become:

$$\mathcal{R}(x,y,z) = \frac{A}{\Delta(z-x)}\left[e^{k(z-y)} - e^{-k(z-y)}\right] \tag{26}$$

$$\mathcal{R}(z,y,x) = \frac{A}{\Delta(z-x)} \left[e^{k(y-x)} - e^{-k(y-x)}\right] \tag{27}$$

$$\mathcal{T}(x,y,z) = \frac{\Delta(z-y)}{\Delta(z-x)} \tag{28}$$

$$\mathcal{T}(z,y,x) = \frac{\Delta(y-x)}{\Delta(z-x)} \tag{29}$$

where we have written:

$$\text{"A"} \quad \text{for} \quad \left(1 - \frac{a^2D^2}{k^2}\right) \tag{30}$$

Example 3: Differential Equations for R and T Factors

The discussions of Sec. 8.6 based the derivation of the R and T factors for irradiance on knowledge of the irradiance field in X(a,b). This procedure is reversed in invariant imbedding theory: The global reflectance and transmittance properties of a medium are found first, and from these, via the imbedding relation, the internal irradiance field is determined. The R and T factors are determined by solving the differential equations they are found to obey. We shall state and discuss these differential equations in this example. The operator versions of these differential equations were derived in Sec. 7.1 and are of a fundamental kind known as *Riccati equations*. The gestalt of these equations for R and T factors is the same for the R and T operators, and as shown in Ref. [251], is independent of the geometric structure of the medium. In the irradiance context the differential equations for the R and T factors take their simplest form, the R and T factors being numerical valued functions rather than function valued functions. In such a context the heuristic manipulations of Sec. 7.3 become fully justified using ordinary calculus.

The derivations of the differential equations of the R and T factors in the two-D theory start with the local and global forms of the principles of invariance for irradiance fields. These principles are reproduced below for convenient comparison. The two main (global) principles of invariance are:

$$\text{I.} \quad H(y,+) = H(z,+)T(z,y) + H(y,-)R(y,z) \tag{31}$$

$$\text{II.} \quad H(y,-) = H(x,-)T(x,y) + H(y,+)R(y,x) \tag{32}$$

The two local forms of the principles of invariance for irradiance are:

$$-\frac{dH(y,+)}{dy} = \tau(y,+)H(y,+) + \rho(y,-)H(y,-) \tag{33}$$

$$\frac{dH(y,-)}{dy} = \tau(y,-)H(y,-) + \rho(y,+)H(y,+) \tag{34}$$

These principles are taken from (1) and (2) of Sec. 8.1 and (6) and (7) of Sec. 8.2, respectively. Adopting the assumptions of two-D theory, the local transmittance and reflectance factors become depth independent (re (8) of Sec. 8.5), so that the "y" can be dropped from the notations "$\tau(y,\pm)$", "$\rho(y,\pm)$".

The derivation procedure now follows that of (10) through (29) of Sec. 7.1 in all its essential respects. Therefore we need only state results, and in forms indigenous to the irradiance context:

$$\text{I}' \quad -\frac{\partial R(x,z)}{\partial x} = \rho(-) + [\tau(+) + \tau(-)]R(x,z) + \rho(+)R^2(x,z) \tag{35}$$

$$\text{II}' \quad \frac{\partial T(x,z)}{\partial z} = [\tau(-) + \rho(-)R(z,x)]T(x,z) \tag{36}$$

$$\text{III}' \quad \frac{\partial R(x,z)}{\partial z} = T(x,z)\rho(-)T(z,x) \tag{37}$$

$$\text{IV}' \quad -\frac{\partial T(x,z)}{\partial x} = [\tau(-) + \rho(+)R(x,z)]T(x,z) \tag{38}$$

The differences between (35) and (38) and their operator correspondents in Sec. 7.1 (starting with (18) of Sec. 7.1) immediately strike the eye: the presence of signed ρ and τ factors and the rearrangement of terms, both in the present set. Both differences are superficial and can be erased with a few strokes of the pen and some accompanying reasons. The signed ρ and τ factors reflect the two-D nature of the light field and summarize the fact that the local optical properties of $X(a,b)$, with respect to irradiance, are accordingly *anisotropic,* as discussed at some length in Sec. 8.5. If, in the developments of Sec. 7.1, we chose to explicitly consider aniostropic media, then the four operators $\rho_{\pm}(y)$, $\tau_{\pm}(y)$ would have been used throughout that discussion (see note after (4) of Sec. 7.1). However, for simplicity and for practical reasons, namely that anisotropic media are seldom encountered in practice, the developments of Sec. 7.1 took their present form. Readers may view the anisotropic versions of the differential equations for general R and T operators in Sec. 25 and Sec. 125 of Ref. [251]; furthermore, Sec. 7.7 and Sec. 7.13 contain differential equations which incorporate signed local reflectance and transmittance operators. As far as the order of the terms within I' through IV' above is concerned, we need only note that we are now working with real valued functions of real numbers rather than with operators, so that commutativity of the present multiplications is in force. As a result we have been able to rearrange the differential equations of R and T to look "more natural" to the eye.

The solutions of (35) through (38), namely $R(x,z)$ and $T(x,z)$ and their companions $R(z,x)$, $T(z,x)$ are readily

obtained. However, we need only note that they are of the form (10) through (13). The advantage of the differential equations (35) through (38) is that they may be integrated even when ρ and τ *vary with depth*, so that they potentially transcend the two-D theory in versatility. The reader should now derive the completely general differential equations for $R(x,z)$ and $T(x,z)$ starting with (1) through (4) and with no assumptions on the homogeneity of $X(a,b)$. The results should be of the form:

$$\text{I}' \quad -\frac{\partial R(x,z)}{\partial x} = \rho(x,-) + [\tau(x,+) + \tau(x,-)]R(x,z) + \rho(x,+)R^2(x,z) \tag{39}$$

$$\text{II}' \quad \frac{\partial T(x,z)}{\partial z} = [\tau(z,-) + \rho(z,-)R(z,x)]T(x,z) \tag{40}$$

$$\text{III}' \quad \frac{\partial R(x,z)}{\partial z} = T(x,z)\rho(z,-)T(z,x) \tag{41}$$

$$\text{IV}' \quad -\frac{\partial T(x,z)}{\partial x} = [\tau(x,-) + \rho(x,+)R(x,z)]T(x,z) \tag{42}$$

Equations (39) and (42) are the key relations here. Using this we can find $R(x,z)$, $T(x,z)$ for $X(x,z)$, with initial conditions $R(z,z) = 0$, $T(z,z) = 1$. The reader should now develop the equations for $R(z,x)$, $T(z,x)$. (See bibliographic notes.)

Example 4: Third Order Semigroup Properties of $\mathcal{R}$ and $\mathcal{T}$ Factors

The semigroup properties of the $\mathcal{R}$ and $\mathcal{T}$ *operators*, as given in (52) and (53) of Sec. 3.7, were some of the most frequently used properties during our discussions of Chapter 7. We have already used their irradiance counterparts in Sec. 8.5 to advance the theory of boundary effects in plane-parallel media (cf. (92) and (93) of Sec. 8.5). We now assemble the full family of semigroup relations which hold among the members of $\Gamma_3(a,b)$ (re: (36) of Sec. 3.7). It will be an instructive exercise to derive these semigroup relations directly from the general semigroup relations stated in (85) through (88) of Sec. 3.7. Equations (85) through (88) of Sec. 3.7 were referred to as the *fourth order semigroup relations* since they are stated for members of $\Gamma_4(a,b)$ (re: (57) of Sec. 3.7). By allowing the depth variables w and x to coincide in the notation "$\Gamma(u,w;v,x)$", we obtain the *third order operators* of $\Gamma_3(a,b)$ and if the variables coalesce in just the right manner, the corresponding *third order semigroup relations* are forthcoming from (85) to (88) of Sec. 3.7. The requisite identifications of variables to go from $\Gamma_4(a,b)$ to $\Gamma_3(a,b)$ are given in (72) through (75) of Sec. 3.7. By means of these types of identifications and the appropriate choices of the variables in the fourth order semigroup relations, the requisite third-order counterparts can be obtained as follows.

Starting with (85) of Sec. 3.7, we set $u = a$, $v = b$. The result is:

$$\mathcal{T}(a,y;b,z) = \mathcal{T}(a,w;b,x)\mathcal{T}(w,y;x,z) + \mathcal{R}(a,x;b,w)\mathcal{R}(x,y;w,z) \ . \tag{43}$$

Now, the next step will seem like so much sorcery to the reader who has not carefully studied Examples 6 and 7 of Sec. 3.7: In the derivations of (69) and (71) of Sec. 3.7, it was discovered that the *right-end variables* of the extended $\mathcal{R}$ and $\mathcal{T}$ operators were "loose" in the sense that:

$$\mathcal{T}(u,w;v,x) = \mathcal{T}(u,w;v,x') \tag{44}$$

$$\mathcal{R}(v,w;u,x) = \mathcal{R}(v,w;u,x') \tag{45}$$

and:

$$\mathcal{T}(v,x;u,w) = \mathcal{T}(v,x;u,w') \tag{46}$$

$$\mathcal{R}(u,x;v,w) = \mathcal{R}(u,x;v,w') \tag{47}$$

for arbitrary values of x, x', and w, w', respectively. We can take advantage of this "looseness' of the right-end variables to write, in (43);

$$\mathcal{T}(a,y;b,z) = \mathcal{T}(a,y;b,y) = \mathcal{T}(a,y,b) \tag{48}$$

Furthermore, on the same basis, setting $w = a$, $x = b$, and using (84) of Sec. 3.7 to note that $\mathcal{T}(a,a;b,b)$ is an identity operator, the first summand in (43) can be exchanged for:

$$\mathcal{T}(a,a;b,b)\ \mathcal{T}(a,y;x,y) = \mathcal{T}(a,y,x) \tag{49}$$

and retaining the identifications $w = a$, $x = b$, the second summand can be exchanged for:

$$\mathcal{R}(a,x;b,x)\mathcal{R}(x,y;a,y) = \mathcal{R}(a,x,b)\mathcal{R}(x,y,a) \tag{50}$$

Assembling these results, (43) becomes:

$$\mathcal{T}(a,y,b) = \mathcal{T}(a,y,x) + \mathcal{R}(a,x,b)\mathcal{R}(x,y,a) \tag{51}$$

The physical interpretation of (51) is clearly discernible with the help of Fig. 8.4. In that figure the depths y and x are reversed from their usual lexicographic order, but that is an accident of notational choices in the present derivation. The essential physical import of (51) is that radiant flux completely transmitted from level a to level y *in the medium* $X(a,b)$ may be thought of as consisting of two parts: that completely transmitted from level a to level y *in the medium* $X(a,x)$, and that completely reflected from the lower boundary of $X(a,x)$ to level y, where the amount of flux incident at the lower boundary of $X(a,x)$ is that completely reflected from level a to level x *in the medium* $X(a,b)$. Complicated, but correct; and something the intuitions of quite experienced radiative transferists or transport theorists

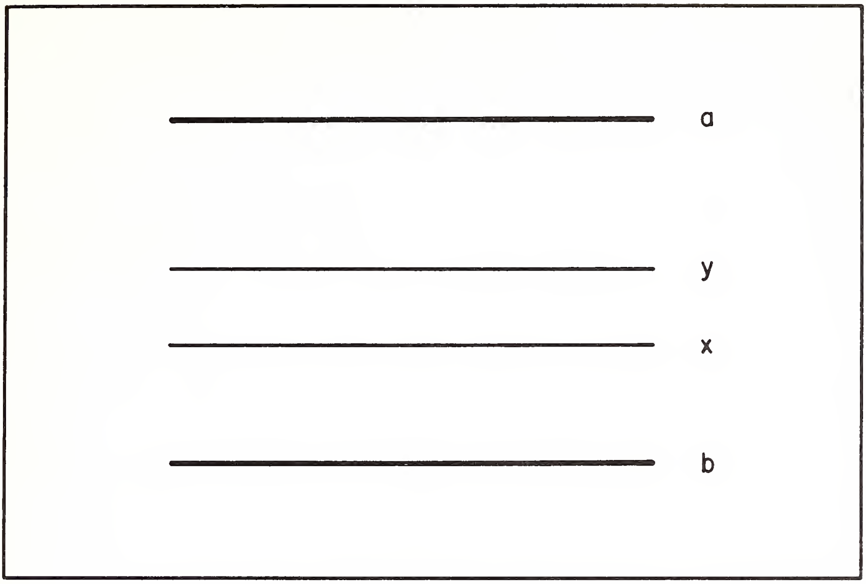


FIG. 8.4 For visualizing equation (51).

could overlook. An equation of the kind (51), but for complete reflectance operators, was found in (69) of Sec. 7.4 in constructing the star product of the invariant imbedding operators.

The reader may now deduce in a similar manner three more semigroup relations from (86) to (88) of Sec. 3.7. These, along with (51), are stated below using the customary lexicographic orders on the depth variables; as shown in Fig. 8.5.

$$\mathcal{T}(a,x,b) = \mathcal{T}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{R}(z,x,a) \tag{52}$$

$$\mathcal{T}(a,z,b) = \mathcal{T}(a,x,b)\mathcal{T}(x,z,b) \tag{53}$$

$$\mathcal{R}(a,x,b) = \mathcal{R}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{T}(z,x,a) \tag{54}$$

$$\mathcal{R}(a,z,b) = \mathcal{T}(a,x,b)\mathcal{R}(x,z,b) \tag{55}$$

Equations (52) through (55) constitute the required complete set of *third-order semigroup relations* for members of $\Gamma_3(a,b)$. Since we have derived them without using any special assumptions on (85) through (88) of Sec. 3.7, the relations, as they stand, hold in any one-parameter optical medium in either the radiance or irradiance contexts. Three of the preceding types of equations have been derived as a matter of course during earlier discussions (69) through (72)

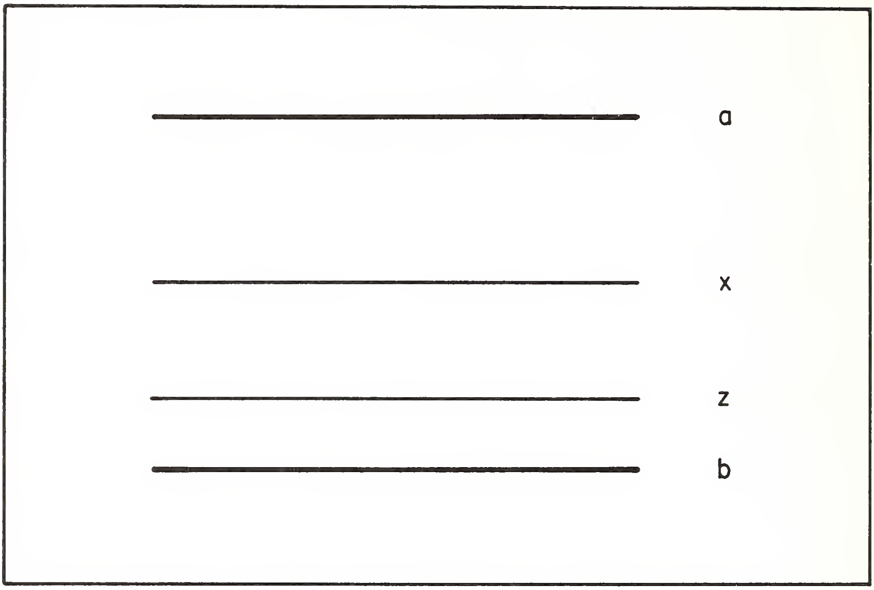


FIG. 8.5 For visualizing equations (52) through (55).

of Sec. 7.4) with (52) as the relative newcomer of the preceding group. Thus we finally arrive at a complete gathering of the third-order semigroup relations. It is clear that a companion group to that of (52) through (55) for upward incident flux follows from (85) to (88) of Sec. 3.7 by reversing the sense of the incident flows in the medium, i.e., by interchanging "a" with "b" and "x" with "z" everywhere in equations (52) through (55) but *not* in Fig. 8.5.

The reader may find it of interest to derive (52) through (55) in still another way so as to gain confidence in their validity. Thus, begin with the three invariant imbedding relations written for the setting of Fig. 8.5:

$$(N_+(z),N_-(z)) = (N_+(b),N_-(a))\mathcal{M}(a,z,b) \tag{56}$$

$$(N_+(x),N_-(x)) = (N_+(b),N_-(a))\mathcal{M}(a,x,b) \tag{57}$$

$$(N_+(x),N_-(x)) = (N_+(z),N_-(a))\mathcal{M}(a,x,z) \tag{58}$$

Then use the representation of $N_+(z)$, as given by (56), in (58). The result is:

$$(N_+(x),N_-(x)) = (N_+(b)\mathcal{T}(b,z,a) + N_-(a)\mathcal{R}(a,z,b),N_-(a))\mathcal{M}(a,x,z)) \tag{59}$$

Equating the right side of (59) with that of (57), we have, e.g.,

$$N_+(x) = N_+(b)\mathcal{T}(b,x,a) + N_-(a)\mathcal{R}(a,x,b)$$

$$= [N_+(b)\mathcal{T}(b,z,a) + N_-(a)\mathcal{R}(a,z,b)]\mathcal{T}(z,x,a) + N_-(a)\mathcal{R}(a,x,z) \ .$$

Since $N_+(b)$ and $N_-(a)$ are arbitrary:

$$\mathcal{R}(a,x,b) = \mathcal{R}(a,x,z) + \mathcal{R}(a,z,b)\mathcal{T}(z,x,a) \quad ,$$

which is (54). The remaining three semigroup relations may be found in a similar manner.

Example 5: Systematic Analyses of Boundary Effects

In this example we amplify and systematize the discussion in Sec. 8.5 concerned with boundary effects on the irradiance field in a plane-parallel medium. The problem we shall consider is analogous to that in Example 6 of Sec. 3.9. However, our main goal now goes beyond the formulation of the boundary effects problem and takes the form of a systematic method of analysis of the problem into its component parts. The main tools we shall use are the third-order semigroup relations of the preceding example. The discussion shall be carried out in a plane-parallel medium for an irradiance field. However, both these special geometric and radiometric settings are readily exchanged for more general settings without any change in the algebraic topography of the results.

To fix ideas, we shall consider a homogeneous plane-parallel medium $X(a,b)$ with a two-D light field, with two reflecting boundaries, X_a and X_b, and a reflecting interface X_y, $a \le y \le b$. Therefore we have a composite optical medium consisting of the union of three surfaces and two slabs. See Fig. 8.6. We shall denote this space by "$X_5(a,b)$". Each of the five parts of $X_5(a,b)$ is generally endowed with a quartet of reflectance-transmittance factors: Those of X_a, X_y, X_b are as developed in Sec. 3.3 and Sec. 3.4. Those of $X(a,y)$ and $X(y,b)$ are as developed in Sec. 3.6 and Sec. 3.7. The resultant roster of interaction operators is as follows:

$$\text{For } X_u: \quad r_\pm(u), t_\pm(u), \tag{60}$$

$$\text{For } X(u,v): \quad \begin{matrix} R(u,v), T(u,v) \\ R(v,u), T(v,u) \end{matrix} \left\{ \begin{matrix} u = a; \ v = y \\ u = y; \ v = b \end{matrix} \right\} \tag{61}$$

Thus there is a total of twelve surface interaction factors and a total of eight slab interaction factors, twenty in all. Now, for the present analysis this relatively large collection of spaces and operators is imagined to be assembled piece by piece starting with surface X_b. To this we add the slab $X(y,b)$ (considered boundaryless) to obtain what we shall refer to as "$X_2(y,b)$". We continue adding pieces until $X_5(a,b)$ is reconstructed. Thus, we shall write:

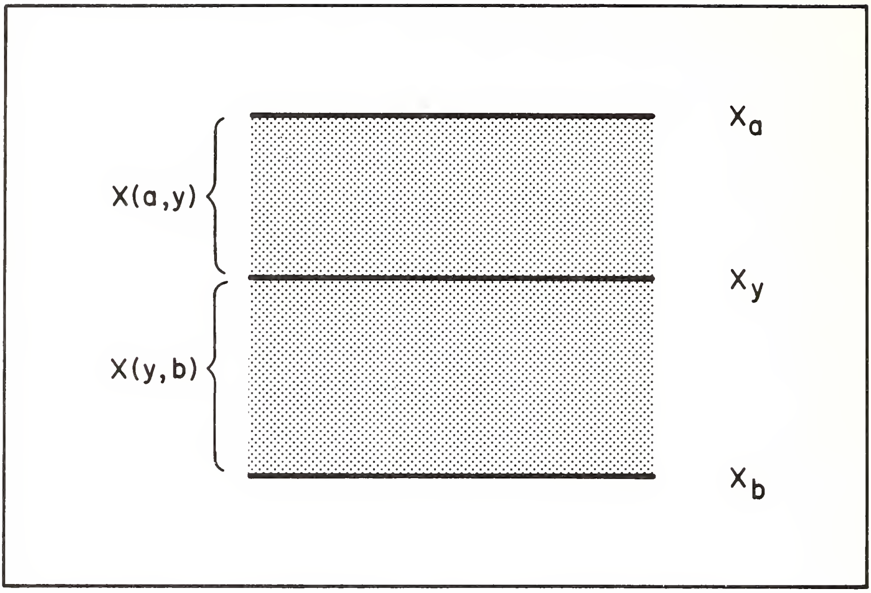


FIG. 8.6 A five-part optical medium consisting of three reflecting-transmitting surfaces (heavy lines) and two diffusing media (dotted).

$$\text{"}X_2(y,b)\text{"} \quad \text{for} \quad X(y,b) \cup X_b \tag{62}$$

$$\text{"}X_3(y,b)\text{"} \quad \text{for} \quad X_y \cup X_2(y,b) \tag{63}$$

$$\text{"}X_4(a,b)\text{"} \quad \text{for} \quad X(a,y) \cup X_3(y,b) \tag{64}$$

This mode of construction of $X_5(a,b)$ is patterned after the invariant imbedding process used in Sec. 7.13 (re: Fig. 7.25). We shall develop a discrete-space approach to the present invariant imbedding process. We are now ready for the analysis.

Stage 1 of the present invariant imbedding process (Fig. 8.7(a)) consists in finding the invariant imbedding operators for X_b. These are simply the factors $r_\pm(b)$, $t_\pm(b)$. To establish a systematic notation which will hold in all stages of the analysis we write:

$$\text{"}\mathcal{R}(b,b,b+1) \quad \text{for} \quad r_-(b) \tag{65}$$

$$\text{"}\mathcal{T}(b-1,b,b)\text{"} \quad \text{for} \quad t_-(b) \tag{66}$$

$$\text{"}\mathcal{R}(b,b,b-1)\text{"} \quad \text{for} \quad r_+(b) \tag{67}$$

$$\text{"}\mathcal{T}(b+1,b,b)\text{"} \quad \text{for} \quad t_+(b) \tag{68}$$

This notation is patterned after that of general discrete

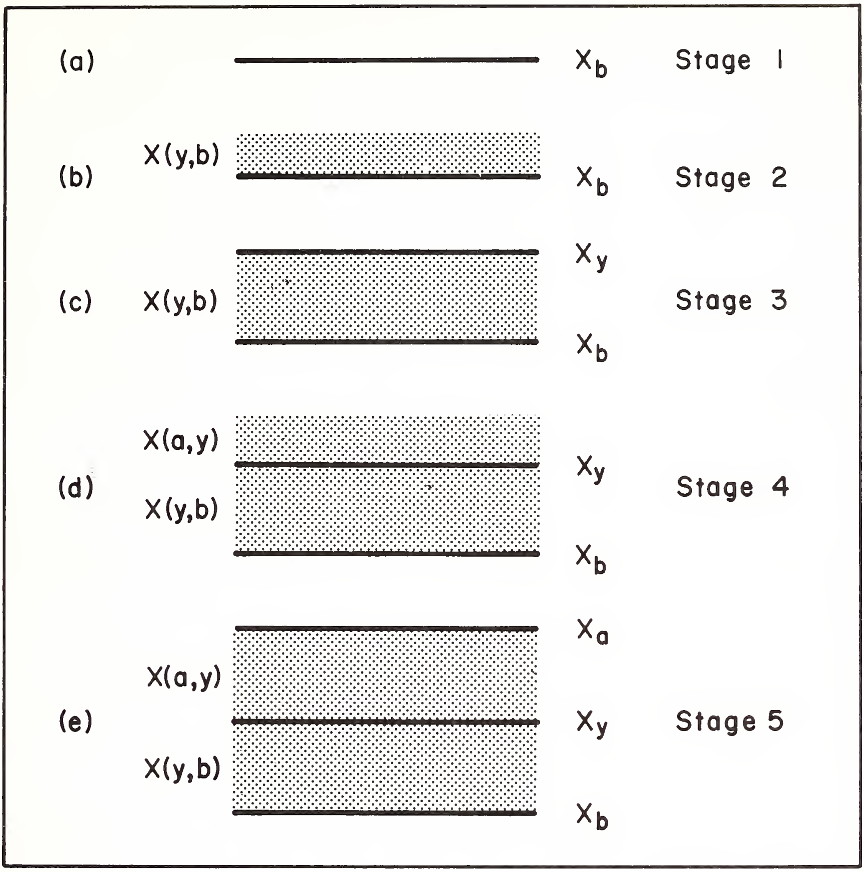


FIG. 8.7 A systematic analysis (and resynthesis) of the medium in Fig. 8.6.

space theory, (Ref. [251]). Hence, for Stage 1, the pertinent invariant imbedding relation associated with X_b is:

$$(H(b,+),H(b,-)) = (H(b+1,+),H(b-1,-))\mathcal{M}(b-1,b,b+1) \quad (69)$$

Here $\mathcal{M}(b-1,b,b+1)$ is the 2×2 matrix made up from the four factors defined in (65) through (68). By the simple notational device of adding 1 to b and subtracting 1 from b, we can conveniently denote the incident irradiances *on* X_b *considered as an isolated medium.* This tactic will be used repeatedly below and is the signal that a discrete-space calculation is in progress.

Stage 2 of the present invariant imbedding process finds the operators for $X_2(y,b)$. The $\mathcal{M}$- operator for $X_2(y,b)$ is $\mathcal{M}(y,z,b+1)$, a 2×2 matrix. Thus, by (52) and (54), adapted to the present one-parameter medium, we have:

$$\mathcal{R}(y,z,b+1) = \mathcal{R}(y,z,b) + \mathcal{R}(y,b,b+1)\mathcal{T}(b,z,y) \tag{70}$$

$$\mathcal{T}(y,z,b+1) = \mathcal{T}(y,z,b) + \mathcal{R}(y,b,b+1)\mathcal{R}(b,z,y) \tag{71}$$

which hold for $a \le y \le z \le b$. These equations represent $\mathcal{R}(y,z,b+1)$ and $\mathcal{T}(y,z,b+1)$, two of the four components of $\mathcal{M}(y,z,b+1)$, in terms of the four complete $\mathcal{R}$ and $\mathcal{T}$ factors for $X(y,b)$ (known) and the operator $\mathcal{R}(y,b,b+1)$. By (55), adapted to the present setting, we have:

$$\mathcal{R}(y,b,b+1) = \mathcal{T}(y,b,b+1)\mathcal{R}(b,b,b+1) \tag{72}$$

$\mathcal{R}(b,b,b+1)$ is known from (65). It remains to find $\mathcal{T}(y,b,b+1)$. At this point (42) of Sec. 3.7 is used to find:

$$\mathcal{T}(y,b,b+1) = T(y,b)[I - R(b,y)R(b,b+1)]^{-1} \tag{73}$$

Here $T(y,b)$ and $R(b,y)$ are known properties of $X(y,b)$, and "$R(b,b+1)$" is another name for $\mathcal{R}(b,b,b+1)$ using the reduction equations (44) through (47) of Sec. 3.7. Therefore (70) through (73) completely analyze two of the operators of $\mathcal{M}(y,z,b+1)$ associated with $X_2(y,b)$.

The remaining two operators, $\mathcal{R}(b+1,z,y)$ and $\mathcal{T}(b+1,z,y)$ in $\mathcal{M}(y,z,b+1)$ are analyzed similarly. Thus, from (53) and (55)

$$\mathcal{R}(b+1,z,y) = \mathcal{T}(b+1,b,y)\mathcal{R}(b,z,y) \tag{74}$$

$$\mathcal{T}(b+1,z,y) = \mathcal{T}(b+1,b,y)\mathcal{T}(b,z,y) \tag{75}$$

which hold for $a \le y \le z \le b$. These equations represent $\mathcal{R}(b+1,z,y)$ and $\mathcal{T}(b+1,z,y)$ in terms of the two complete $\mathcal{R}$ and $\mathcal{T}$ operators for $X(y,b)$ (known) and the operator $\mathcal{T}(b+1,b,y)$. By (43) of Sec. 3.7 we have:

$$\mathcal{T}(b+1,b,y) = T(b+1,b)[I - R(b,y)R(b,b+1)]^{-1}$$

Here $R(b,y)$ is a known property of $X(y,b)$ and "$T(b+1,b)$" and "$R(b,b+1)$" are other names for $t_+(b)$ and $r_-(b)$, respectively. We summarize the constructions of Stage 2 by the following equation:

$$\mathcal{M}(y,z,b+1) = \begin{bmatrix} \mathcal{T}(b+1,b,y)\mathcal{T}(b,z,y) & \mathcal{T}(b+1,b,y)\mathcal{R}(b,z,y) \\ \mathcal{R}(y,z,b)+\mathcal{R}(y,b,b+1)\mathcal{T}(b,z,y) & \mathcal{T}(y,z,b)+\mathcal{R}(y,b,b+1)\mathcal{R}(b,z,y) \end{bmatrix}$$

The preceding matrix can be analyzed into factors of the invariant imbedding type as follows:

$$\boxed{\begin{aligned} \mathcal{M}(y,z,b+1) = [C_- + \mathcal{M}(y,b,b+1)C_+]\mathcal{M}(y,z,b) \\ a \le y \le z \le b \end{aligned}} \tag{77}$$

where C_+ and C_- are defined in (4) and (5) of Sec. 7.4. Now $I_\pm$ occurring in $C_\pm$ are simply the number 1 in the present irradiance context. Equation (77) shows how the invariant imbedding operators of two contiguous media can be algebraically combined to yield the invariant imbedding operator for their union. In this case the media are the slab $X(y,b)$ and the surface X_b.

Stage 3 of the present invariant imbedding process finds the $\mathcal{M}$-operator for $X_3(y,b)$. The $\mathcal{M}$-operator for $X_3(y,b)$ is the 2×2 matrix $\mathcal{M}(y-1,z,b+1)$. Thus, by (53), and (55), adapted to the present one-parameter medium, we have:

$$\mathcal{R}(y-1,z,b+1) = \mathcal{T}(y-1,y,b+1)\mathcal{R}(y,z,b+1) \tag{78}$$

$$\mathcal{T}(y-1,z,b+1) = \mathcal{T}(y-1,y,b+1)\mathcal{T}(y,z,b+1) \tag{79}$$

which hold for $y \le z \le b$. The two factors $\mathcal{R}(y,z,b+1)$ and $\mathcal{T}(y,z,b+1)$ are known from the preceding stage of the analysis. The remaining factor is obtained by means of (42) of Sec. 3.7:

$$\mathcal{T}(y-1,y,b+1) = T(y-1,y)[I - R(y,b+1)R(y,y-1)]^{-1} \tag{80}$$

Here "$T(y-1,y)$" and "$R(y,y-1)$" are simply other names for $t_-(y)$ and $r_+(y)$, respectively, used in discrete-space theory. Finally,

$$R(y,b+1) = \mathcal{R}(y,y,b+1) \quad , \tag{81}$$

which is known from the preceding stage of analysis. The remaining two components of $\mathcal{M}(y-1,z,b+1)$ are found by means of (52) and (54);

$$\mathcal{R}(b+1,z,y-1) = \mathcal{R}(b+1,z,y) + \mathcal{R}(b+1,y,y-1)\mathcal{T}(y,z,b+1) \tag{82}$$

$$\mathcal{T}(b+1,z,y-1) = \mathcal{T}(b+1,z,y) + \mathcal{R}(b+1,y,y-1)\mathcal{R}(y,z,b+1) \tag{83}$$

which holds for $y \le z \le b$. Here all factors are known from Stage 2 except for $\mathcal{R}(b+1,y,y-1)$, which can be obtained using the semigroup relation:

$$\mathcal{R}(b+1,y,y-1) = \mathcal{T}(b+1,y,y-1)\mathcal{R}(y,y,y-1)$$

Finally $\mathcal{T}(b+1,y,y-1)$ is found by means of (43) of Sec. 3.7.

Stage 3 may be summarized by the following equation:

$$\mathcal{M}(y-1,z,b+1) =$$

$$= \begin{bmatrix} \mathcal{T}(b+1,z,y)+\mathcal{R}(b+1,y,y-1)\mathcal{R}(y,z,b+1) & \mathcal{R}(b+1,z,y)+\mathcal{R}(b+1,y,y-1)\mathcal{T}(y,z,b+1) \\ \mathcal{T}(y-1,y,b+1)\mathcal{R}(y,z,b+1) & \mathcal{T}(y-1,y,b+1)\mathcal{T}(y,z,b+1) \end{bmatrix}$$

In other words:

$$\boxed{\begin{gathered}\mathcal{M}(y-1,z,b+1) = [C_+ + \mathcal{M}(y-1,y,b+1)C_-]\mathcal{M}(y,z,b+1) \\ y \le z \le b\end{gathered}} \tag{84}$$

The pattern that is forming is now sufficiently clear so that the reader may complete stages 4 and 5. Observe that (84), as it stands, solves the general boundary-effect problem of a medium $X(y,b)$ with interreflecting boundaries X_y and X_b.

Example 6: Invariant Imbedding Operators for Interacting Media

We consider next the problem of predicting the irradiance field within a medium $X(a,c)$ composed of two contiguous media $X(a,b)$, $X(b,c)$. The media may be of infinite depth or they may be degenerate, i.e., they may be surfaces. For example, if $X(a,b)$ is degenerate, then we write "$X(a-1,a)$" for $X(a,b)$ and construct the reflectances and transmittances in the manner described in Example 5 above. If there is an interface at level b, let it belong to $X(a,b)$. Our main purpose in this example is to present a unified algebraic approach to the problem of irradiance (or any other* radiometric) fields in sets of contiguous plane-parallel media. We have developed sufficient background for the solution of this problem in the preceding Examples 4 and 5 and in Sec. 8.5 (re: (94) of that section) to permit the broad algebraic techniques of the present example to be followed without difficulty.

Figure 8.8 depicts the composite medium $X(a,c) = X(a,b) \cup X(b,c)$. We assume that the operators (2×2 matrices) $\mathcal{M}(a,y,b)$ and $\mathcal{M}(b,y,c)$ associated with the component media $X(a,b)$, $X(b,c)$ are known. Our goal is to characterize $\mathcal{M}(a,y,c)$, the invariant imbedding operator for $X(a,c)$, in terms of the operators $\mathcal{M}(a,y,b)$ and $\mathcal{M}(b,y,c)$. The analysis of the problem reduces to the two cases, depicted in Fig. 8.8. Consider case (a). The irradiance field $(H(y,+), H(y,-))$ at depth y, $a \le y \le b$ may be viewed from two vantage points: as an irradiance field in $X(a,b)$, which is the response of the isolated medium $X(a,b)$ to the incident irradiances $H(b,+)$, $H(a,-)$; or as an irradiance field in $X(a,c)$ which is the response of $X(a,c)$ to the incident irradiances $H(c,+)$, $H(a,-)$. The first interpretation is represented as:

---

*In the event that any of the present results are adapted to the *radiance* context, and media with different indices of refraction are considered, it will be understood that each radiance will be divided by the square of the index of refraction of the medium to which it pertains (cf. (4) of Sec. 7.6), so that we use N/n^2 rather than simply N throughout any formula.

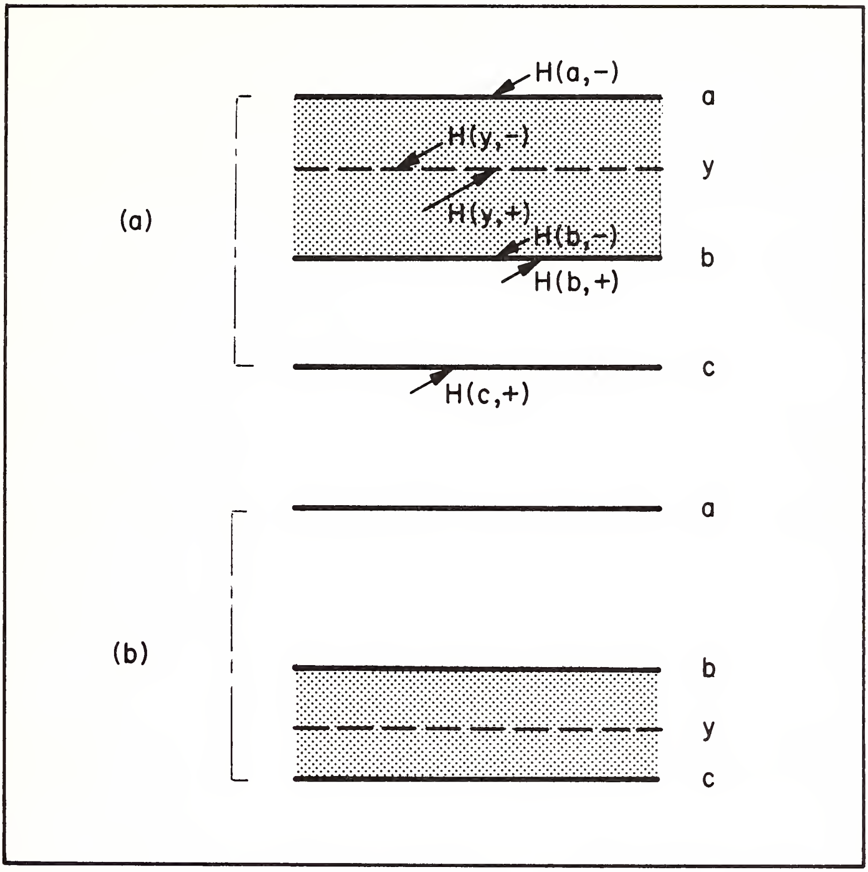


FIG. 8.8 The response of the composite medium X(a,c) can be characterized algebraically in terms of the responses of its components X(a,b) and X(b,c).

$$(H(y,+),H(y,-)) = (H(b,+),H(a,-))\mathcal{M}(a,y,b) \qquad (85)$$

The second interpretation is represented as:

$$(H(y,+),H(y,-)) = (H(c,+),H(a,-))\mathcal{M}(a,y,c) \qquad (86)$$

Finally, the irradiance field (H(b,+),H(b,-)) may be represented as:

$$(H(b,+),H(b,-)) = (H(c,+),H(a,-))\mathcal{M}(a,b,c) \qquad (87)$$

Using the contracting matrices C_+, C_- in (4) and (5) of Sec. 7.4, it follows from (87) that:

$$(H(b,+),0) = (H(c,+),H(a,-))\mathcal{M}(a,b,c)C_+$$

Further we have:

$$(0,H(a,-)) = (H(c,+),H(a,-))C_-$$

Adding these two equations, we obtain:

$$(H(b,+),H(a,-)) = (H(c,+),H(a,-))[C_- + \mathcal{M}(a,b,c)C_+] \tag{88}$$

which, operated on by $\mathcal{M}(a,y,b)$, would, according to (85), yield an alternate representation of $(H(y,+),H(y,-))$ to that given in (86). Hence, since $(H(c,+),H(a,-))$ is arbitrary,

$$\boxed{\begin{array}{c}\mathcal{M}(a,y,c) = [C_- + \mathcal{M}(a,b,c)C_+]\mathcal{M}(a,y,b)\\ a \le y \le b \le c\end{array}} \tag{89}$$

Case (b) in Fig. 8.8 proceeds analogously, and the form of $\mathcal{M}(a,y,c)$ in this case turns out to be:

$$\boxed{\begin{array}{c}\mathcal{M}(a,y,c) = [C_+ + \mathcal{M}(a,b,c)C_-]\mathcal{M}(b,y,c)\\ a \le b \le y \le c\end{array}} \tag{90}$$

Equation (89) is the general form of (77) and (90) that of (84). Further, (89) and (90) implicitly contain the third-order semigroup relations (52) through (55) for the settings of Fig. 8.8.

The generality of (89) and (90) resides in the possibility of either component $X(a,b)$ or $X(b,c)$ being itself a composite space made up any number of contiguous slabs and internal boundaries (see, e.g., Examples 3 and 4 of Sec. 3.4). Equations (89) and (90) serve as guides in the construction of $\mathcal{M}(a,y,c)$, knowing the corresponding operators for its component spaces. Hence (89) and (90) constitute an *inductive step* in the construction of $\mathcal{M}(a,y,c)$ in precise analogy to the customary inductive step used in the application of the principle of induction in mathematical arguments, where one proceeds from a statement $P(n)$ associated with an integer n to statement $P(n+1)$. One interesting application of (89) and (90) would be to work out the complete details of assigning distinct pairs $D_i(\pm)$ of distribution values to each slab $X(a_i,a_{i+1})$ in a sequence of n slabs comprising $X(a,b)$, so that (6) through (9) may be used in actual numerical calculations based on (89) and (90).

It remains to observe how $\mathcal{M}(a,b,c)$ is found. The reader may have noted, on the basis of the discussion in

Example 5, that equations (40) through (43) of Sec. 3.7 will serve adequately in this task. In this connection, we observe that an elegant algebraic formulation of (40) through (43) of Sec. 3.7 is possible by using the star product for $\Gamma_3(a,b)$ introduced in (75) of Sec. 7.4. For by (76) of Sec. 7.4 we have:

$$\mathcal{M}(a,b,c) = \mathcal{M}(a,b,b) * \mathcal{M}(b,b,c) \qquad (91)$$

Continuing on this algebraic level, we can further resolve $\mathcal{M}(a,b,b)$ and $\mathcal{M}(b,b,c)$ by means of the M-operators (2) of Sec. 7.4. It is easy to see that:

$$\mathcal{M}(a,b,b) = \begin{bmatrix} I & R(b,a) \\ 0 & T(a,b) \end{bmatrix} = [C_+ + M(a,b)C_-] \quad ; \qquad (92)$$

$$\mathcal{M}(b,b,c) = \begin{bmatrix} T(c,b) & 0 \\ R(b,c) & I \end{bmatrix} = [M(b,c)C_+ + C_-] \quad . \qquad (93)$$

Finally, the star product * for $G_2(a,b)$, as defined in (35) of Sec. 7.4 and studied in (37) and (38) of that section, may be used analogously to (91) to find the R and T operators of the union of contiguous media. Thus, e.g., the equation:

$$M(a,c) = M(a,b) * M(b,c) \qquad (94)$$

shows how to algebraically construct the standard R and T factors for X(a,c), knowing those for X(a,b) and X(b,c).

Example 7: Differential Equations Governing $\mathcal{R}$ and $\mathcal{T}$ Factors

The preceding examples have shown the key role played by the complete $\mathcal{R}$ and $\mathcal{T}$ factors in the determination of the irradiance field in a variety of practical instances. It is of interest to observe that these $\mathcal{R}$ and $\mathcal{T}$ factors may be obtained directly by integrating the differential equation that governs them. Several such differential equations were developed in (11) through (13) of Sec. 7.5. The one we select for attention here is (38) of Sec. 7.5:

$$\boxed{\frac{d\mathcal{A}(y)}{dy} = \mathcal{A}(y)\mathcal{K}(y)} \qquad (95)$$

defined for each y in the depth interval [a , b] and with initial conditions:

$$\mathcal{Q}(b) = [0, T(a,b)] \tag{96}$$

or:

$$\mathcal{Q}(a) = [R(a,b), I] \tag{97}$$

Here we have written:

$$\text{"}\mathcal{Q}(y)\text{"} \quad \text{for} \quad (\mathcal{R}(a,y,b), \mathcal{T}(a,y,b)) \quad ,$$

and $\mathcal{K}(y)$ now has the form given in (9) of Sec. 8.2. For example, knowing $\mathcal{K}(y)$ and R(a,b), one can find $\mathcal{R}(a,y,b)$, $\mathcal{T}(a,y,b)$ directly by integrating (95) from a to y. The integrations may be theoretical or numerical, where appropriate. Further observations on $\mathcal{Q}(y)$, of interest to the irradiance context, are given in (11) through (15) of Sec. 7.10.

Example 8: Method of Modules for Irradiance Fields

We now present an illustration of the method of modules, as developed in Sec. 7.8, for the case of irradiance fields. Equations (14) of Sec. 7.8 are readily put to use in irradiance computations once $\mathcal{T}(d)$ is obtained. From (8) and (21) we have at once:

$$\begin{aligned}
\mathcal{T}(d) &= \lim_{z\to\infty} \mathcal{T}(0,d,z) \\
&= \lim_{z\to\infty} \frac{\Delta(d,z)}{\Delta(0,z)} \\
&= \lim_{z\to\infty} \frac{\Delta(z-d)}{\Delta(z)}\, e^{+(k_+ + k_-)d} \\
&= e^{k_- d}
\end{aligned} \tag{98}$$

Equations (14) of Sec. 7.8 then take the form

$$\boxed{\begin{aligned}
H(jd,+) &= H(0,-)e^{jk_- d}\, R_\infty(-) \qquad (99)\\
H(jd,-) &= H(0,-)e^{jk_- d} \qquad (100)
\end{aligned}}$$

where we have written:

$$\text{"}R_\infty(-)\text{"} \quad \text{for} \quad \lim_{z\to\infty} R(0,z) \tag{101}$$

and which, by (10), has the representation:

$$R_\infty(-) = \frac{g_-(+)}{g_-(-)} = \frac{1 + \frac{a(-)}{k_-}}{1 - \frac{a(+)}{k_-}} = \frac{k_- + a(-)}{k_- - a(+)} \tag{102}$$

The latter equalities follow from (11) of Sec. 8.5.

Example 9: Method of Semigroups for Irradiance Fields

The method of semigroups, applied to general radiance fields in Sec. 7.9, yields formulas for $H(y,\pm)$ in $X(0,\infty)$. Thus from (10) and (12) of Sec. 7.9, we now may write:

$$\boxed{H(y,-) = H(0,-) \exp\{ (\tau(-) + \rho(+)R_\infty(-))y \}} \tag{103}$$

and as usual:

$$H(y,+) = H(y,-)R_\infty(-) \tag{104}$$

The present setting is $X(0,\infty)$ and we have used the two-D theory concepts $\tau(-)$, $\rho(+)$, and $R_\infty(-)$. The latter constant was defined in (101). In view of (99) (which holds for arbitrary d, so let $d = y$, thereby fixing j as 1) we see that we must have:

$$\boxed{k_- = \tau(-) + \rho(+)R_\infty(-)} \tag{105}$$

It may be verified that this is consistent with (102). This is the desired connection between k_-, $\tau(-)$, $\rho(+)$, and $R_\infty(-)$. Equations (102) and (105) are the first views we have of an important set of general connections which hold between the theoretical and the exact observable counterparts to these concepts, and which are studied in detail in Sec. 9.2.

Example 10: Irradiance Fields Generated by Internal Sources

We devote tne final example of this section to illustrating some of the relations developed in Sec. 7.3 for internal sources in the special case of irradiance field (i.e., one-dimensional) settings. As a result we gain perspective on the structure of the one-D and two-D moels for internal sources discussed in general in (44) through (66) of Sec. 8.5, and in particular in (25) of Sec. 8.6.

According to the general conversion principle, stated in the introductory remarks to this section, every functional equation developed in Sec. 7.13 may be converted to the present irradiance context. Because of this we shall devote most of the discussion to the task of bringing to light some special relations for internal-source generated irradiance fields which hold *only* in the irradiance context.

Toward this end, consider the observations in (87) through (92) of Sec. 7.13 concerned with the asymmetry of the Ψ-operator. It was observed that the invariant imbedding operators, such as $\mathcal{R}(s,y,b)$ and $\mathcal{T}(s,y,b)$ were generally distinct from their duals $\mathcal{R}^+(y,s,b)$ and $\mathcal{T}^+(y,s,b)$, respectively. An examination of the matter showed that if $\mathcal{R}(s,y,b)$ and $\mathcal{R}^+(y,s,b)$ were ever equal in some setting, then the computation details of the internal source problem in that setting would be effectively halved with respect to the general case. Therefore a search for a reciprocity property between $\mathcal{R}(s,y,b)$ and $\mathcal{R}^+(s,y,b)$ was launched. It did not require an extensive analysis to see that the requisite reciprocity property (i.e., the equality of $\mathcal{R}(s,y,b)$ and $\mathcal{R}^+(y,s,b)$) is generally barred by polarity of the R and T operators and general noncommutativity of the integral operators. Since commutativity of $\mathcal{R}$ and $\mathcal{T}$ factors is now available, and since the R and T factors of one-D models do not possess polarity (re: (35) and (36) of Sec. 8.5, and also (24) and (25),) we return to the matter of reciprocity of $\mathcal{R}(s,y,b)$ and $\mathcal{R}^+(y,s,b)$ and reexamine some of the functional relations of Sec. 7.13 in the present simpler setting. Therefore for the remainder of this example, we shall work in a separable plane-parallel medium X(a,b) in which the one-D assumptions of Sec. 8.6 are in force. First we explicitly verify that:

$$\mathcal{R}(s,y,b) = \mathcal{R}^\dagger(y,s,b) \tag{106}$$

$$\mathcal{T}(s,y,b) = \mathcal{T}^\dagger(y,s,b) \tag{107}$$

Figure 8.9 is a schematic summary of the relative location of levels s,y,b in X(a,b), pertinent to (106) and (107), Thus we have $a \le s < y \le b$. Furthermore, the entities of (106) and (107) may be seen in their correct place in a general invariant imbedding process on X(a,b) by referring to stage 2 of Fig. 7.25. Now, to test the validity of (106) we use (1) and (25) of Sec. 7.13 to write down the equivalent statement to (106):

$$T(s,y)\Psi_{-+}(y,y:a,b) = \Psi_{-+}(y,y:a,b)T(y,s) \quad .$$

By commutativity of numerical multiplication we have:

$$T(s,y)\Psi_{-+}(y,y:a,b) = T(y,s)\ \Psi_{-+}(y,y:a,b) \quad .$$

This statement can be generally true only in a one-D setting, for since $\Psi_{-+}(y,y:a,b) \neq 0$ for all $a < y < b$, we require:

$$T(s,y) = T(y,s) \quad ,$$

which holds generally only for one-D irradiance fields (re: (24) and (25)).

As a consequence of (106) and (107), and two more equations of the same kind:

$$\mathcal{R}(b,y,s) = \mathcal{R}^\dagger(y,b,s) \tag{108}$$

$$\mathcal{T}(b,y,s) = \mathcal{T}^\dagger(y,b,s) \quad , \tag{109}$$

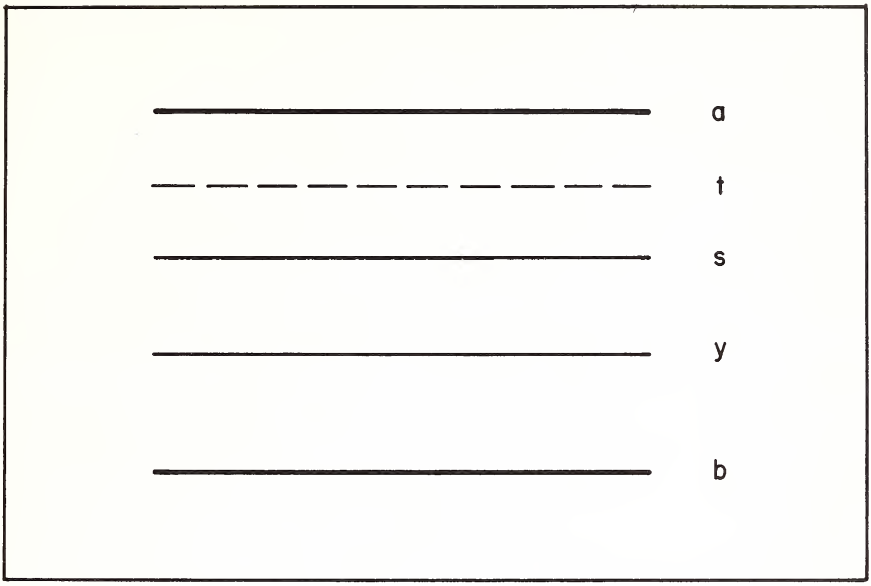


FIG. 8.9 An internal-source setting for irradiance fields. The sources are at level s, the observation level is at y. This setting falls under case 4, stage 3 of the general invariant imbedding scheme in Fig. 7.25.

we see that the number of invariant imbedding factors for X(a,b) is reduced by a factor of two.

The physical significance of equations (106) through (109) is of interest in the present discussions and we pause to make it clear. Consider (106) and Fig. 8.9. The left side of (106), namely the complete reflectance factor $\mathcal{R}(s,y,b)$, describes the amount of upward irradiance at level y in X(s,b) induced by a unit downward irradiance at level s . On the other side, the dual reflectance factor $\mathcal{R}^{\dagger}(y,s,b)$ describes the amount of upward irradiance at level s in X(s,b) induced by a unit downward irradiance at level y . The present geometric setting (plane-parallel medium) and physical assumptions for that setting (separability and isotropy of X(a,b)) combine to imply the reciprocity statement (106). The remaining three reciprocity statements are interpreted similarly.

We are now ready to illustrate some of the functional relations for internal-source generated light fields, as developed in Sec. 7.13. Equation (86) of Sec. 7.13 is one of the general results of that section and, since its associated geometric setting is precisely that of Fig. 8.9, we select it for illustration.

The first part of the illustration consists in casting (86) of Sec. 7.13 into $\mathcal{m}$-operator notation, in order to emphasize the fact that stage 3 of an invariant imbedding

process (Fig. 7.25) can be carried out using the $\mathcal{M}$-operators constructed in stage 2. Furthermore, the $\mathcal{M}$-operator notation is simpler and of more intuitive value to the present discussion. The conversion to $\mathcal{M}$-operators is effected by means of (63) and (69) of Sec. 7.13:

$$\Psi(s,y:s,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(s,y,b) & \mathcal{T}(s,y,b) \end{bmatrix} = C_{-}\mathcal{M}(s,y,b) \quad (110)$$

$$\Psi(s,t:t,b) = \begin{bmatrix} \mathcal{T}^{\dagger}(s,t,b) & 0 \\ \mathcal{R}^{\dagger}(s,t,b) & 0 \end{bmatrix} = \mathcal{M}^{\dagger}(s,t,b)C_{+} \quad (111)$$

$$\Psi(t,y:t,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(t,y,b) & \mathcal{T}(t,y,b) \end{bmatrix} = C_{-}\mathcal{M}(t,y,b) \quad (112)$$

It follows that (86) of Sec. 7.13 may be written:

$$\Psi(s,y:a,b) = C_{-}\mathcal{M}(s,y,b) + \int_a^s \mathcal{M}^{\dagger}(s,t,b)C_{+}\mathcal{K}(t)C_{-}\mathcal{M}(t,y,b)\,dt \quad (113)$$

or, directly in matrix form:

$$\Psi(s,y:a,b) = \begin{bmatrix} 0 & 0 \\ \mathcal{R}(s,y,b) & \mathcal{T}(s,y,b) \end{bmatrix} +$$

$$+ \int_a^s \begin{bmatrix} \mathcal{T}^{\dagger}(s,t,b) & 0 \\ \mathcal{R}^{\dagger}(s,t,b) & 0 \end{bmatrix} \begin{bmatrix} -\tau & \rho \\ -\rho & \tau \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathcal{R}(t,y,b) & \mathcal{T}(t,y,b) \end{bmatrix} dt$$

$$(114)$$

in which $\mathcal{K}(t)$ is now in its appropriate form for one-D irradiance theory, i.e., $\tau(y,\pm)$ and $\rho(y,\pm)$ are independent of y and $\pm$, (cf. (9) of Sec. 8.2; (11) and (12) of Sec. 8.3; and (2) through (6) of Sec. 8.6).

In order to illustrate (114) in its simplest form, we next assume that the medium $X(a,b)$ is infinitely deep. Thus, we assume $a = 0$ and $b = \infty$. The corresponding forms of the complete $\mathcal{R}$ and $\mathcal{T}$ factors are obtained from (26) and (28). From (26):

$$\mathcal{R}(s,y,\infty) = \lim_{b\to\infty}\mathcal{R}(s,y,b) = \lim_{b\to\infty} \frac{A}{\Delta(b-s)}\left[e^{k(b-y)} - e^{-k(b-y)}\right]$$
$$= R_\infty\, e^{-k(y-s)} \quad . \tag{115}$$

From (28):

$$\mathcal{T}(s,y,\infty) = \lim_{b\to\infty}\mathcal{T}(s,y,b) = \lim_{b\to\infty} \frac{\Delta(b-y)}{\Delta(b-s)} = e^{-k(y-s)} \quad . \tag{116}$$

Hence:

$$\mathcal{R}(s,y,\infty) = R_\infty \mathcal{T}(s,y,\infty) \quad . \tag{117}$$

Using (115) through (117) in (114) and recalling (106) through (109), the result is:

$$\Psi(s,y:0,\infty) = e^{-k(y-s)}\begin{bmatrix} 0 & 0 \\ R_\infty & 1 \end{bmatrix} +$$
$$+\left[\int_0^s e^{2kt}dt\right] \rho e^{-k(s+y)}\begin{bmatrix} R_\infty & 1 \\ R_\infty^2 & R_\infty \end{bmatrix} \tag{118}$$

There are many ways to arrange the final form of (118). For example, one such form is:

$$\boxed{\Psi(s,y:0,\infty) = e^{-k(y-s)}\begin{bmatrix} 0 & 0 \\ R_\infty & 1 \end{bmatrix} + \frac{\rho}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right]\begin{bmatrix} R_\infty & 1 \\ R_\infty^2 & R_\infty \end{bmatrix}} \tag{119}$$

From this form we can pick off the four components of $\Psi(s,y:0,\infty)$:

$$\Psi_{++}(s,y:0,\infty) = \frac{\rho R_\infty}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{120}$$

$$\Psi_{+-}(s,y:0,\infty) = \frac{\rho}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{121}$$

$$\Psi_{-+}(s,y:0,\infty) = R_\infty\, e^{-k(y-s)} + \frac{\rho R_\infty^2}{2k}\cdot$$
$$\cdot\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \tag{122}$$

$$\Psi_{--}(s,y:0,\infty) = e^{-k(y-s)} + \frac{\rho R_\infty}{2k}\left[e^{-k(y-s)} - e^{-k(s+y)}\right] \qquad (123)$$

From these, in turn, we find the local Ψ - factors for one-D irradiance fields in $X(0,\infty)$:

$$\Psi_{++}(s,s:0,\infty) = \frac{\rho R_\infty}{2k}\left[1 - e^{-2ks}\right] \qquad (124)$$

$$\Psi_{+-}(s,s:0,\infty) = \frac{\rho}{2k}\left[1 - e^{-2ks}\right] \qquad (125)$$

$$\Psi_{-+}(s,s:0,\infty) = R_\infty\left[1 + \frac{\rho R_\infty}{2k}(1 - e^{-2ks})\right] \qquad (126)$$

$$\Psi_{--}(s,s:0,\infty) = \frac{\rho R_\infty}{2k}\left[1 - e^{-2ks}\right] \qquad (127)$$

The preceding set of eight equations offers an excellent opportunity to illustrate the general functional relations (31) through (34) of Sec. 3.9 and especially their complementary relations (39) through (42) of Sec. 7.13. Furthermore, the relations (6) through (15) of Sec. 7.13 are also illustrated in perhaps their simplest settings. Observe, for example, how (39) of 7.13 guided the derivation of (127) from (123), and how (14) of Sec. 7.13 was used to find (126).

When the source level s is allowed to sink lower and lower into $X(0,\infty)$, equations (124) through (127) go to the relatively simple forms:

$$\lim_{s\to\infty} \Psi_{++}(s,s:0,\infty) = \frac{\rho R_\infty}{2k} \qquad (128)$$

$$\lim_{s\to\infty} \Psi_{+-}(s,s:0,\infty) = \frac{\rho}{2k} \qquad (129)$$

$$\lim_{s\to\infty} \Psi_{-+}(s,s:0,\infty) = R_\infty\left[1 + \frac{\rho R_\infty}{2k}\right] \qquad (130)$$

$$\lim_{s\to\infty} \Psi_{--}(s,s:0,\infty) = \frac{\rho R_\infty}{2k} \qquad (131)$$

Furthermore, if both source level s and observation level y descend into $X(0,\infty)$ so that the difference $d = y-s$ remains fixed, the observed boundary effects at level 0 eventually die away and (120) through (123) yield:

$$\lim_{s\to\infty} \Psi_{++}(s,s+d:0,\infty) = \frac{\rho R_\infty}{2k}\, e^{-kd} \qquad (132)$$

$$\lim_{s\to\infty} \Psi_{+-}(s,s+d:0,\infty) = \frac{\rho}{2k}\, e^{-kd} \qquad (133)$$

$$\lim_{s\to\infty} \Psi_{-+}(s,s+d:0,\infty) = R_\infty e^{-kd}\left[1 + \frac{\rho R_\infty}{2k}\right] \tag{134}$$

$$\lim_{s\to\infty} \Psi_{--}(s,s+d:0,\infty) = e^{-kd}\left[1 + \frac{\rho R_\infty}{2k}\right] \tag{135}$$

An unexpected dividend accrues from the preceding array of Ψ-factor relations. Observe first that (128)-(131) agree with our intuitive ideas that the relation between the source irradiance $H^\circ(s,+)$ and the response field $H(s,+)$, should be the same as that between $H^\circ(s,-)$ and $H(s,-)$ when boundaries are far away from level s (because the medium is optically symmetric about very deep levels). By the same intuitive expectations, Ψ_{+-} and Ψ_{-+} of (126) and (130) should be numerically equal. Apparently, this does not seem to be the case. However, if we rely on the correctness of our principles and algebraic manipulations, to yield up (129) and (130) then on the basis of our intuition we are led to conclude that:

$$\frac{\rho}{2k} = R_\infty\left[1 + \frac{\rho R_\infty}{2k}\right] ,$$

or equivalently that:

$$\boxed{\frac{\rho}{2k} = \frac{R_\infty}{1 - R_\infty^2}} \tag{136}$$

where ρ is the local reflectance (i.e., backscattering) factor for the one-D theory and k and R_∞ are respectively the decay rate and reflectance factor associated with irradiance fields in $X(0,\infty)$. It follows that all the preceding results (128)-(131) may be characterized in terms of R_∞ only. The reader may establish (136) independently of the preceding argument by using the connections (105) above, with (32) of Sec. 7.3, which holds in the irradiance context also. Still further connections between k, ρ, R_∞ and related concepts are available in Chapters 9 and 10.

8.8 A Model for Vector Irradiance Fields

The purpose of this section is to apply the vector theory of the irradiance field to an important class of scattering-absorbing optical media, namely the class of natural hydrosols consisting, e.g., of oceans, harbors, and lakes. The application is of practical value in that it yields explicit expressions for the depth-dependence of the irradiance vector in terms of its components at the surface and certain of the optical properties of these media. Furthermore, the discussion presents particularly simple interpretations of the quasi-potential and related functions, arising in vector analysis and which are pertinent to the description of natural light fields. These vector interpretations emerge naturally

from the geometry and physics of the present application, and will be given as the discussion proceeds. In this way we add to the evidence that the formalism of the "photic field" as developed by Moon, Spencer, and others (Refs. [187], [188]) is of more than academic interest, and in fact provides an elegant tool for the study of the light vector $\mathbf{H}(s)$ in the practical settings encountered in the study of hydrologic optics.

While the practical context of the present discussion is limited specifically to that of natural hydrosols, the mathematical arguments apply equally well to any arbitrary plane-parallel scattering-absorbing medium in which the light vector possesses a quasi-potential. The radiometric prerequisites for the present discussion are given in (2)-(16) of Sec. 2.8. A useful text on vector analysis is Ref. [30].

The Quasi-Irrotational Light Field in Natural Waters

We fix the stage of the present discussion by adopting a stratified plane-parallel medium $X(0,b)$ with stratified light field. The present discussion makes use of the concept of a quasi-irrotational light field, i.e., a light field in which at each depth z of a natural hydrosol, the irradiance vector satisifes the condition:

$$\mathbf{H}(z) \cdot \left[\nabla \times \mathbf{H}(z) \right] = 0 \tag{1}$$

In general, $\mathbf{H}$, the *vector-irradiance function*, (or *light field*), is defined at each point x (which stands for the ordered triple (x,y,z)) of an optical medium by writing:

$$\text{"}\mathbf{H}(x)\text{"} \quad \text{for} \quad \int_{\Xi} \xi N(x,\xi)\, d\Omega(\xi) \quad , \tag{2}$$

where Ξ is the unit sphere (the collection of all unit vectors) in euclidean three-space E_3, and $N(x,\cdot)$ is the radiance distribution at point x (see Sec. 2.8).

As will be shown in (12) below, the justification of the use of the relation (1) rests on the following two vectorial versions of well-known experimental facts about the spatial and directional distribution of light in natural hydrosols:

(i) For every fixed $z \geq 0$ in $X(0,b)$, $\mathbf{H}(x,y,z)$ is independent of x and y.

(ii) For every fixed pair (x,y), $\mathbf{H}(x,y,z)$ lies in a fixed vertical plane for all $z \geq 0$.

Of course, some variations of $\mathbf{H}$ on horizontal planes, and some oscillations of the vertical plane containing $\mathbf{H}$ do occur in all natural hydrosols. However, properties (i) and (ii) summarize the two most readily apparent permanent gross features of the light field in natural waters, on which it is possible to develop a mathematical theory of the light vector $\mathbf{H}(x)$.

Interpretations of the Integrating Factor

Since our interests lie principally in the physical and geometrical aspects of natural light fields, it would be instructive to develop some physical interpretations of (1) and concepts immediately related to it. This we now do.

The general theory of vector fields asserts that to each quasi-irrotational light field one may associate two real-valued functions Φ and ζ, defined on the appropriate subset of $X(0,z_1)$ representing the optical medium. These functions have the property that:

$$\mathbf{H}(x) = \frac{1}{\zeta(x)} \nabla \Phi(x) \quad . \tag{3}$$

Φ is the *quasi-potential function,* and ζ is the *integrating factor*, unique to within a multiplicative constant, associated with Φ. Equation (3) is the necessary and sufficient condition that:

$$\mathbf{H}(x) \cdot [\nabla \times \mathbf{H}(x)] = 0$$

at each x of the medium. (See, e.g., Sec. 105, Ref. [30].)

In the present context the function ζ has particularly simple and interesting geometrical and physical interpretations. We begin with the geometric interpretation.

Figure 8.10 defines a terrestrially based coordinate system usually adopted for the discussion of the light fields in natural hydrosols (re: Sec. 2.4). The fixed plane referred to in property (ii) is taken as the xz-plane, and thus lies in the plane of the figure. The standard unit vectors $\mathbf{i}$ and $\mathbf{k}$ are positioned as shown. The unit vector $\mathbf{j}$ along the positive y-axis is normal to the plane of the figure and directed away from the reader.

Consider an arbitrary rectangular path ABCD in the xz-plane such that its sides are parallel to the coordinate axes. According to (3) and properties (i) and (ii) of the light field in natural hydrosols, it follows that:

$$\int_{ABCD} \zeta(x)\mathbf{H}(x) \cdot d\mathbf{s} = 0 \quad ,$$

so that:

$$\int_{AB} \zeta(x)\mathbf{H}(z_1) \cdot ds = \int_{DC} \zeta(x)\mathbf{H}(z_2) \cdot d\mathbf{s}$$

The condition (i) suggests that ζ can be independent of x and y. By (13) of Sec. 2.8:

$$\overline{H}(z,\mathbf{i}) = \mathbf{H}(z) \cdot \mathbf{i}$$

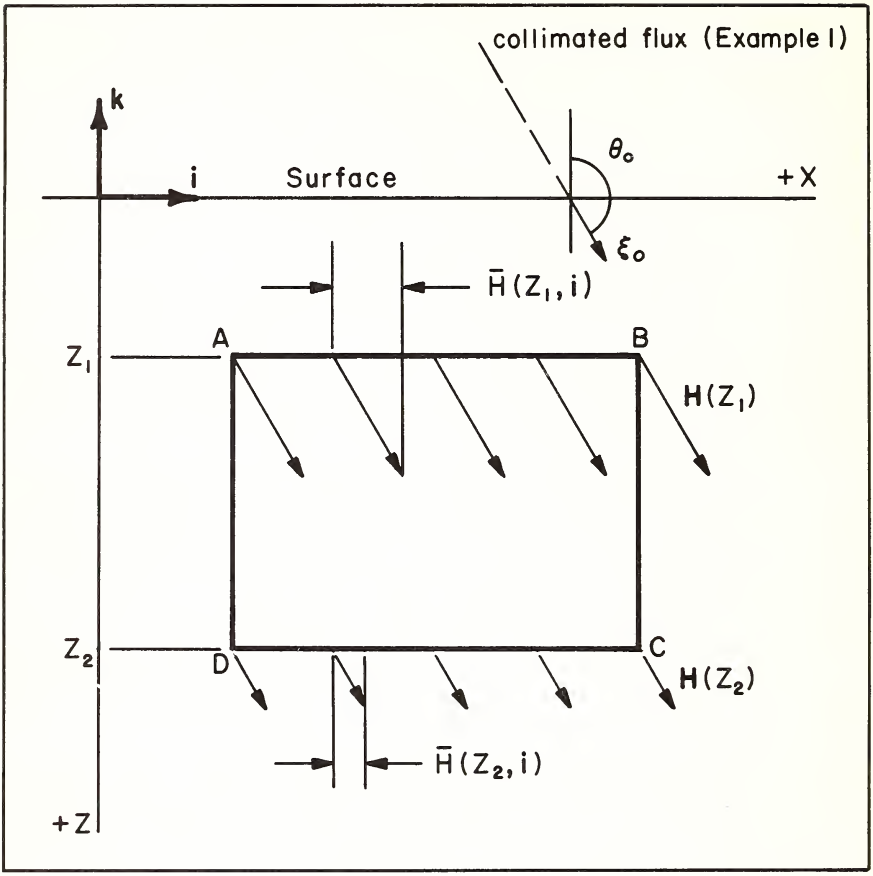


FIG. 8.10 How to visualize a quasi-irrotational irradiance vector field.

so that the integral equality above implies:

$$\zeta(z_1)\bar{H}(z_1,\mathbf{i}) = \zeta(z_2)\bar{H}(z_2,\mathbf{i}) \quad . \tag{4}$$

From this and the fact that ζ is determined only to within a multiplicative constant, we can set $\zeta(0) = 1$ and so:

$$\zeta(z)\bar{H}(z,\mathbf{i}) = \bar{H}(0,\mathbf{i}) \quad , \tag{5}$$

for all z in $[0,b]$. Thus ζ may be selected as a dimensionless quantity which stretches the horizontal component $\bar{H}(z,\mathbf{i})$ to $\bar{H}(0,\mathbf{i})$ at every depth z in $X(0,b)$.

Next we consider the physical interpretation of $\zeta(z)$. The invariance with depth of the product:

$$\zeta(z)\bar{H}(z,\mathbf{i})$$

shows that the depth dependence of $\zeta(z)$ is such that its logarithmic derivative is equal, to within an algebraic sign, to the logarithmic derivative of the net horizontal irradiance $\bar{H}(z,\mathbf{i})$. That is:

$$\frac{1}{\zeta(z)} \cdot \frac{d\zeta(z)}{dz} = \frac{-1}{\bar{H}(z,\mathbf{i})} \cdot \frac{d\bar{H}(z,\mathbf{i})}{dz} \quad . \tag{6}$$

This logarithmic derivative of $\bar{H}(z,\mathbf{i})$ will be denoted by "$\bar{K}(z,\mathbf{i})$". Now the logarithmic derivative $\bar{K}(z,\mathbf{i})$ is but one member of an important family of apparent optical properties used in modern hydrologic optics, as we shall see in Chapter 9. This family of optical properties includes such well-known quantities as $k(z)$, where:

$$k(z) = -\frac{1}{h(z)}\frac{dh(z)}{dz} \tag{7}$$

and where $h(z)$ is scalar irradiance of depth z :

$$h(z) = \int_{\Xi} N(z,\xi)\,d\Omega(\xi) \tag{8}$$

Thus, in the terminology of Chapter 9, the logarithmic derivative of ζ is none other than the *k-function* for the net horizontal irradiance in the $\mathbf{i}$-direction. According to (6) we may represent $\zeta(z)$ as:

$$\zeta(z) = e^{C(z)} \tag{9}$$

where we have written:

$$\text{"}C(z)\text{"} \quad \text{for} \quad \int_0^z \bar{K}(z',\mathbf{i})\,dz' \tag{10}$$

The Curl and Divergence of the Submarine Light Field

The vectorial concepts of curl and divergence have been applied very often in hydrodynamic theory, and other field theories. However, there is also a useful role for these concepts in the description of light fields in natural hydrosols. Under the present assumptions (i) and (ii) about the light field, and with the adopted coordinate system, the curl of $\mathbf{H}$ may be verified to be of the form:

$$\nabla\times\mathbf{H}(z) = -\,\mathbf{j}\,\frac{d\bar{H}(z,\mathbf{i})}{dz}$$

so that:

$$\boxed{\nabla\times\mathbf{H}(z) = \mathbf{j}\bar{K}(z,\mathbf{i})\bar{H}(z,\mathbf{i})} \tag{11}$$

It follows from this and property (ii) that:

$$\boxed{\mathbf{H}(z) \cdot [\nabla\times\mathbf{H}(z)] = 0} \tag{12}$$

The derivation of the divergence relation for the light field in (source-free) scattering-absorbing media will require the use of the equation of transfer for radiance:

$$\xi \cdot \nabla N(z,\xi) = -\alpha(x)N(x,\xi) + N_*(z,\xi) \quad , \tag{13}$$

where the path function N_* is given by:

$$N_*(x,\xi) = \int_\Xi N(x,\xi')\sigma(x;\xi';\xi)d\Omega(\xi') \tag{14}$$

Here σ and α are, respectively, the volume scattering function and the volume attentuation function.

Returning to (13) and integrating each side over Ξ, we have:

$$\nabla \cdot \mathbf{H}(x) = -\alpha(x)h(x) + s(x)h(x) \quad ,$$

which reduces to the required divergence relation:

$$\boxed{\nabla \cdot \mathbf{H}(x) = -a(x)h(x)} \tag{15}$$

by using the facts that:

$$s(x) = \int_\Xi \sigma(x;\xi;\xi')d\Omega(\xi') \tag{16}$$

and that:

$$\alpha(x) = a(x) + s(x) \quad , \tag{17}$$

and which were formally introduced in (3) and (4) of Sec. 4.2.

For the present geometry (15) reduces to:

$$\boxed{\frac{d\overline{H}(z,\mathbf{k})}{dz} = a(z)h(z)} \tag{18}$$

where

$$\overline{H}(z,\mathbf{k}) = \mathbf{H}(z) \cdot \mathbf{k} \tag{19}$$

is the vertical component (the net upward irradiance) of the light vector at depth x. (Observe that the derivatives of $\nabla \cdot \mathbf{H}$ are along the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and recall that z is measured positive in the direction $-\mathbf{k}$.) If the medium

X(0,b) has sources, then the term $h_\eta(z)$ is added to the right side of (18). The reader will find it instructive to return to (62) of Sec. 8.4 and view it in the light of (15). Furthermore, it should now be clear that by adding together the equations of (19) of Sec. 8.3, we obtain (18).

General Representation of the Submarine Light Field

The starting point of the present derivation is taken as theorem 9 of Ref. [187] which, applied to the present concepts of radiative transfer in scattering-absorbing media, states that if:

$$\nabla \cdot \mathbf{H}(z) = - a(z)h(z)$$

and:

$$\mathbf{H}(z) \cdot [\nabla\times\mathbf{H}(z)] = 0$$

then a quasi-potential Φ and integrating factor ζ exist such that:

$$\nabla^2\Phi(x,y,z) = \frac{1}{\zeta(z)}\left[- a(z)h(z) + \nabla\left(\frac{1}{\zeta(z)}\right) \cdot \nabla\Phi(x,y,z)\right] \quad (20)$$

Furthermore, it follows from properties (i) and (ii) and the preceding equation that Φ can be at most linear in x and y. We can use this fact along with (3):

$$\mathbf{H}(z) = \frac{1}{\zeta(z)}\left[\mathbf{i}\,\frac{\partial\Phi}{\partial x} + \mathbf{j}\,\frac{\partial\Phi}{\partial y} + \mathbf{k}\,\frac{\partial\Phi}{\partial z}\right] \quad (21)$$

to deduce the requisite representation of $\mathbf{H}(z)$. Now, since Φ is at most linear in x:

$$\frac{\partial\Phi}{\partial x} = A \quad , \quad (22)$$

where A is a constant. Furthermore, since Φ is at most linear in y and by virtue of the present coordinate system we can set:

$$\frac{\partial\Phi}{\partial y} = 0 \quad (23)$$

Hence Φ may be represented in the present context as follows:

$$\Phi(x,y,z) = Ax + f(z) \quad .$$

According to (21) it remains to determine A and $f'(z)$ ($= df(z)/dz$). Toward this end, equation (20) may be written:

$$f''(z) - \overline{K}(z,\mathbf{i}) - \zeta(z)a(z)h(z) = 0 \quad .$$

The integrating factor for this differential equation is clearly

$$\frac{1}{\zeta(z)} = e^{-C(z)} \quad ,$$

so that:

$$(f'(z)/\zeta(z))' = a(z)h(z) \quad ,$$

and:

$$f'(z)/\zeta(z) = f'(0) + \int_0^z a(z')h(z')dz' \quad .$$

Finally, from (21) it follows that:

$$\boxed{\mathbf{H}(z) = \mathbf{i}\bar{H}(0,\mathbf{i})e^{-C(z)} + \mathbf{k}\left[\bar{H}(0,\mathbf{k}) + \int_0^z a(z')h(z')dz'\right]} \tag{24}$$

which is the desired general representation of the vector irradiance function $\mathbf{H}$.

Example 1: The case of Isotropic Scattering

For the remaining portion of this section we will assume that the medium X(0,b) is homogeneous, i.e., that α (and hence σ , s, and a) is independent of depth z . The present example will be concerned with an illustration of the particular form that $\mathbf{H}(z)$ takes in a medium that scatters isotropically and which is irradiated at the upper boundary by collimated flux incident at an angle $\theta_o = -\operatorname{arc}\cos\mu_o$, $\phi_o = 0$, and where ϕ_o is an azimuth angle measured in a horizontal plane from the positive x-axis. The number μ_o is defined as in (70) of Sec. 8.5.

Now it is easy to verify that the diffuse component of the light field (i.e., that part consisting of all radiant flux scattered one or more times) is symmetrical about the z-axis. Hence the net horizontal irradiance receives no contribution from the diffuse light field. Therefore:

$$\bar{H}(z,\mathbf{i}) = \bar{H}(0,\mathbf{i})e^{-\alpha z/\mu_o} \tag{25}$$

so that $\bar{K}(z,\mathbf{i})$ in this case is represented by:

$$\bar{K}(z,\mathbf{i}) = \alpha/\mu_o \tag{26}$$

and C(z) in the general theory above reduces to:

$$C(z) = \alpha z/\mu_o \tag{27}$$

Example 2: Asymptotic Form of the Light Field

In optically infinitely deep media (i.e., $b = \infty$ in $X(0,b)$) the values $k(z)$ of the function k defined in (7) rapidly approach, with increasing z, a fixed magnitude k which is independent of the external lighting conditions and which depends only on the inherent optical properties of the medium. This and related facts we shall explore in some detail in Chapter 10. In view of this fact it is permissible, for most engineering calculations, to assume that there is a depth $z_o \geq 0$ below which $k(z) = k$. From the divergence relation (18) we see that in general:

$$\bar{H}(z_2,\mathbf{k}) - \bar{H}(z_1,\mathbf{k}) = \int_{z_1}^{z_2} a(z)h(z)\,dz \tag{28}$$

so that in particular:

$$\bar{H}(z,\mathbf{k}) - \bar{H}(0,\mathbf{k}) = \int_0^z a(z')h(z')dz'$$

Furthermore, in the present case:

$$\bar{H}(z,\mathbf{k}) - \bar{H}(z_o,\mathbf{k}) = a\int_{z_o}^{z} h(z')dz' = ah(z_o)\int_{z_o}^{z} e^{-k(z'-z_o)}dz'$$

$$= \frac{-\,ah(z_o)e^{kz_o}}{k}\left[e^{-kz} - e^{-kz_o}\right]$$

$$= \frac{ah(z_o)}{k}\left[1 - e^{-k(z-z_o)}\right]$$

It follows that the $\mathbf{k}$-component of $\mathbf{H}(z)$ in (24) may be written:

$$\bar{H}(z,\mathbf{k}) = \bar{H}(0,\mathbf{k}) + \int_0^z a(z')h(z')dz'$$

$$= \bar{H}(z_o,\mathbf{k}) + \int_{z_o}^{z} a(z')h(z')dz'$$

$$= \bar{H}(z_o,\mathbf{k}) + \frac{ah(z_o)}{k}\left[1 - e^{-k(z-z_o)}\right]$$

Now since $\bar{H}(z,\mathbf{k}) \to 0$ as $z \to \infty$ (see, e.g., the two-D models in Secs. 8.5 and 8.6), it follows that we may set:

$$\bar{H}(z_o,\mathbf{k}) = -\left(\frac{a}{k}\right) h(z_o) \quad ,$$

so that (24) reduces to:

$$\mathbf{H}(z) = \mathbf{i}\bar{H}(z_o,\mathbf{i})e^{-C(z-z_o)} + \mathbf{k}\bar{H}(z_o,\mathbf{k})e^{-k(z-z_o)} , \quad (29)$$

for $z \geq z_o$. A further simplification is effected if we can find a suitable approximation for the function C. Thus observe that for $z \geq z_o$, the diffuse component of the light field is essentially symmetrical about the z-axis (Chapter 10) so that, as in the isotropic case (27):

$$\bar{K}(z,\mathbf{i}) = D^o\alpha$$

where we have

$$D^o = \frac{1}{\mu_o}$$

for clear sunny skies with sun at $\theta_o = -\arccos \mu_o$ from the zenith, or for some fixed D^o, with

$$1 \leq D^o \leq 2$$

for overcast days (see (2) of Sec. 8.5). With these assumptions, (24) takes the particularly simple approximate form:

$$\boxed{\mathbf{H}(z) = \mathbf{i}\bar{H}(z_o,\mathbf{i})e^{-D^o\alpha(z-z_o)} + \mathbf{k}\bar{H}(z_o,\mathbf{k})e^{-k(z-z_o)}} , \quad (30)$$

where D^o may take any of the above special values. For most engineering applications it is permissible to take $z_o = 0$ in (29) or (30).

We conclude this example by making a few observations on the limiting directions of $\mathbf{H}$ as $z \to \infty$. First, if $s \neq 0$, then $k < \alpha$, (see, e.g., (9) of Sec. 6.6). If, in addition, we also have $a \neq 0$ then from (30) and the fact that $\bar{H}(a,\mathbf{k})<0$, it is clear that:

$$\lim_{z\to\infty} \frac{\mathbf{H}(z)}{|\mathbf{H}(z)|} = -\mathbf{k} \quad .$$

If, on the other hand, we have $a = 0$, then $\bar{H}(z,\mathbf{k}) = 0$ and:

$$\frac{\mathbf{H}(z)}{|\mathbf{H}(z)|} = \pm\,\mathbf{i}$$

for all z; the direction being that of $\bar{H}(0,\mathbf{i})$. Finally, if $s = 0$, then the problem of the explicit determination of $\mathbf{H}(z)$ for all z reduces to a relatively trivial (although sometimes tedious) calculation. In this case the limiting direction of $\mathbf{H}$ depends in a simple way only on the external lighting conditions. If $N^o(0,\xi)$ represents the incident radiance distribution at $z = 0$, suppose $|\xi_o \cdot \mathbf{k}|$ is the largest value for which $N(0,\xi_o) \neq 0$. Then the limiting direction of $\mathbf{H}(z)$, as $z \to \infty$, is along the line defined by ξ_o.

Global Properties of the Irradiance Field

The curl and divergence of the irradiance field $\mathbf{H}(x)$ show how the field behaves in the neighborhood of a point. In other words, the curl and divergence of $\mathbf{H}(x)$ are *local properties* of $\mathbf{H}(x)$. An interesting and important global property associated with the irradiance field comes from an application of the divergence theorem to (15) and we shall now derive that property.

Let X be an arbitrary connected, bounded homogeneous subset of X(0,b), with boundary surface S and with steady light field. Then on the one hand from (15):

$$\begin{aligned}\int_X \nabla \cdot \mathbf{H}(x)dV(x) &= - \int_X ah(x)dV(x)\\ &= - av \int_X u(x)dV(x)\\ &= - avU(X) \quad . \qquad (31)\end{aligned}$$

The latter two equalities rest on (4) and (12) of Sec. 2.7. On the other hand if $\mathbf{n}(x)$ is the unit inward normal to S at x , then by the divergence theorem, we have:

$$\int_X \nabla \cdot \mathbf{H}(x)dV(x) = - \int_S \mathbf{H}(x) \cdot \mathbf{n}(x)dA(x) \qquad (32)$$

and we shall denote the integral on the right side by "$\overline{P}(S,-)$" which thereby represents the *net inward radiant flux across S* (cf. (8) and (9) of Sec. 2.8). Combining (31) and (32) we have:

$$\boxed{\overline{P}(S,-) = avU(X)} \qquad (33)$$

This equation shows how the net inward flux $\overline{P}(S,-)$ across the boundary of S is related to the energy content U(X) of X and the volume absorption coefficient a of X . Equation (33) may have some practical interest in laboratory procedures of determining the volume absorption coefficient a . The radiant energy content U(X) of X may be obtained by systematically probing X with an instrument which measures scalar irradiance h(x). By numerically integrating the measured values h(x) throughout X , the term vU(X) may be obtained. Further, the term $\overline{P}(S,-)$ may be obtained by traversing the boundary S of X with a subtracting janus plate (re: Sec. 2.8) to find $\overline{H}(x,\mathbf{n}(x))$, and then integrating the measured values. It is clear that this method would be independent of the directional structure of the light field within X . This

fact may be used in laboratory setups to prearrange the light field so as to require a minimal amount of measuring throughout X. If this can be achieved, novel and simple means of determining the volume absorption coefficient will thereby be attained.

8.9 Canonical Representation of Irradiance Fields

We close this chapter on models for irradiance fields with a derivation which parallels the canonical representation of the radiance field given in (5) of Sec. 4.5. It is possible to derive the requisite relation so as to be a proper generalization of (5) of Sec. 4.5, and we shall now follow such a course.

Let X be an arbitrary optical medium. Let x be an arbitrary point of X and to x associate a direction $\mathbf{n}(x)$ and a set $\Xi_0(x)$ of directions. Let us write:

$$\text{"}\mathbf{H}(x,\Xi_0(x))\text{"} \quad \text{for} \quad \int_{\Xi_0(x)} N(x,\xi)\,\xi\, d\Omega(\xi) \quad . \tag{34}$$

This is a generalization of the irradiance vector $\mathbf{H}(x)$. The latter is obtained by requiring $\Xi_0(x) = \Xi$ (cf. (2) of Sec. 2.8, and also (4) of Sec. 2.4 for the numerical instance of (34); and (41) of Sec. 8.6 for an alternate version of (34)). Further, let us write:

$$\text{"}H(x,\mathbf{n}(x),\Xi_0(x))\text{"} \quad \text{or} \quad \text{"}H(x,\mathbf{n},\Xi_0)\text{"} \quad \text{for} \quad \mathbf{n}(x)\cdot\mathbf{H}(x,\Xi_0(x)) \tag{35}$$

It is clear that $H(x,\mathbf{n},\Xi_0)$ is the quantity measured by a subtracting janus plate (Sec. 2.8) whose collecting surfaces are exposed to the set $\Xi_0(x)$ of directions and whose pointer is directed along $\mathbf{n}$ (cf. Figs. 8.11 and 2.21). Associated with $H(x,\mathbf{n},\Xi_0)$ is the scalar irradiance $h(x,\Xi_0)$, where we have written:

$$\text{"}h(x,\Xi_0)\text{"} \quad \text{for} \quad \int_{\Xi_0} N(x,\xi)\, d\Omega(\xi) \tag{36}$$

$h(x,\Xi_0)$ is measured in practice by a spherical irradiance collector exposed to the direction set Ξ_0.

We pause to observe that by suitable choice of Ξ_0, $H(x,\mathbf{n},\Xi_0)$ can generate the usual irradiances $H(x,\xi)$ and the radiances $N(x,\xi)$ (see Fig. 8.11). In the former case we need only set $\mathbf{n}(x) = \xi$ and $\Xi_0(x) = \Xi(\xi)$. In the latter case, we let $\Xi_0(x)$ be a variable circular conical set with central direction ξ. Then:

$$N(x,\xi) = \lim_{\Xi_0 \to \{\xi\}} \frac{H(x,\xi,\Xi_0(x))}{\Omega(\Xi_0)} \tag{37}$$

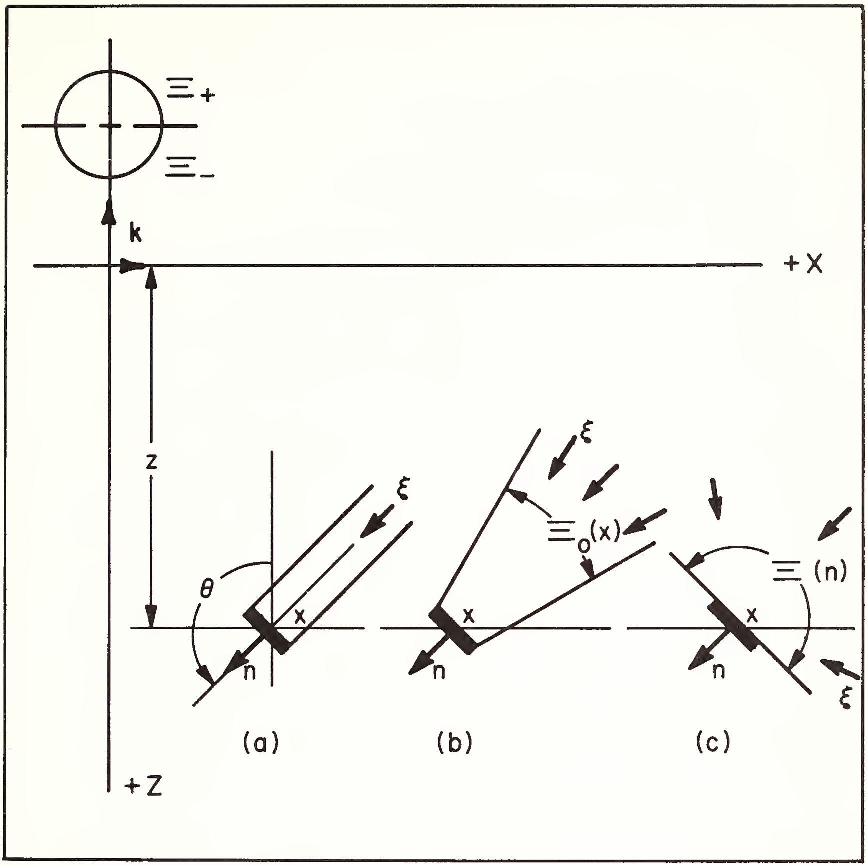


FIG. 8.11 The radiometric constructs, as defined operationally, which are used in the canonical equation for irradiance.

The mathematical basis for this rests in (4) of Sec. 2.5.

Next we define the distribution function associated with the set Ξ_o. We write:

$$\text{"}D(x,\mathbf{n}(x),\Xi_o(x)\text{"} \quad \text{for} \quad \frac{h(x,\Xi_o(x))}{H(x,\mathbf{n}(x),\Xi_o(x))} \tag{38}$$

This is a generalization of the distribution functions introduced in (5) of Sec. 8.3. For example, $D(x,+)$ is obtained from (38) by letting $\mathbf{k} = \mathbf{n}(x)$ and $\Xi_o(x) = \Xi_+$.

We are now ready to cast the equation of transfer into the canonical form for $H(x,\mathbf{n},\Xi_o)$. The derivation model we shall follow is that of (1)-(5) of Sec. 4.5. Thus, let us write:

$$\text{"}K(x,\mathbf{n}(x),\Xi_o(x))\text{"} \quad \text{for} \quad -\frac{1}{H(x,\mathbf{n},\Xi_o)}\,\nabla\cdot\mathbf{H}(x,\Xi_o) \tag{39}$$

Integrating the equation of transfer:

$$\xi\cdot\nabla N(x,\xi) = -\,\alpha(x)N(x,\xi) + \int_{\Xi} N(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

over $\Xi_o(x)$, we have:

$$\nabla\cdot\mathbf{H}(x,\Xi_o) = -\,\alpha(x)h(x,\Xi_o) + \int_{\Xi_o} N_*(x,\xi)\,d\Omega(\xi)$$

and using the preceding definitions, we have:

$$\boxed{K(x,\mathbf{n},\Xi_o) = \alpha(x)D(x,\mathbf{n},\Xi_o) - \frac{1}{H(x,\mathbf{n},\Xi_o)}\int_{\Xi_o} N_*(x,\xi)\,d\Omega(\xi)} \tag{40}$$

A final arrangement yields:

$$\boxed{D(x,\mathbf{n},\Xi_o)H(x,\mathbf{n},\Xi_o) = \frac{\displaystyle\int_{\Xi_o} N_*(x,\xi)\,d\Omega(\xi)}{\left[\alpha(x) - \dfrac{K(x,\mathbf{n},\Xi_o)}{D(x,\mathbf{n},\Xi_o)}\right]}} \tag{41}$$

which is the desired canonical representation of $H(x,\mathbf{n},\Xi_o)$. We readily verify that (41) is a proper generalization of (5) of Sec. 4.5 by recalling (37) and observing that:

$$\lim_{\Xi_o\to\{\xi\}} \frac{\displaystyle\int_{\Xi_o} N_*(x,\xi)\,d\Omega(\xi)}{\Omega(\Xi_o)} = N_*(x,\xi)$$

$$\lim_{\Xi_o\to\{\xi\}} D(x,\xi,\Xi_o) = 1 \quad .$$

8.10 Bibliographic Notes for Chapter 8

It is generally agreed that the history of the two-flow irradiance equations begins with the classic paper by Schuster [279]. The differential equations derived dealt with a pair of irradiance functions representing two counter-flowing streams of radiant energy (one outward and one inward) in a stellar atmosphere. In the hands of Schwarzschild [281], King [138], and Milne [180], Schuster's approach was developed into a relatively complete description of the light field by means of the equation of transfer for radiance. With the advent of the work of Hopf [111], the problems of radiative transfer theory took a deeper mathematical turn, and with the physical insight of Ambarzumian [1], and the industry of Chandrasekhar [43], the notions of the principles of invariance were conceived and exhaustively developed for the simplest plane-parallel settings; and radiative transfer theory as it is known today took its early definitive form.

On the other hand, there followed from Schuster's work another chain of studies which dwelled almost exclusively on his original pair of equations for irradiance; reshaping them, generalizing them, occasionally rediscovering them, and applying them to all manners of optical media from paint and paper, to the atmosphere and the sea. These took, for the most part, the form of the one-D model of Sec. 8.6. The industrial researchers and the geophysicists took alternate turns in the formulations and applications, the results being typified by papers of Schmidt [273], Benford [18], Kubelka and Munk [146], Channon, Renwick, and Storr [44], Mecke [174], Dietzius [65], Silberstein [285], Ryde and Cooper [70], and Duntley [69]. Concurrently, certain Russian authors, notably Gurevic [102], Boldyrev and Alexandrov [27], and Gershun [97], made important contributions to Schuster's theory. The latter papers are curious mixtures of the archaic forms of the equations during that period along with a few isolated brilliant innovations which only much later came into widespread use. Indeed, in the papers of Gurevic [102], and Schmidt [273], for example, may be found the rudimentary but recognizable germ of the idea behind equations (40) and (41) of Sec. 8.7. The fundamental Riccati equation (39) of Sec. 8.7 did not appear in its full form, but as a recognizable primitive variant, and with the physical significances of the coefficient functions being obscure. Stokes [291] also obtained Riccati-type equations in his early researches. The invariant imbedding formulas, of a faint but noticeable variety, can be traced back to Fresnel [94].

With the formulation of neutron diffusion problems there arose a certain amount of mutually profitable cross-fertilization of techniques between neutron diffusion and radiative transfer theories, which stems principally from the papers of Wick [319], and Chandrasekhar [42]. The ramifications of this interaction may be traced in neutron transport theory in [62]. In the Wick paper and subsequently in Chandrasekhar's work, the Schuster two-flow equations were extended to handle n-flows with particular emphasis on the form of the coefficients most suitable to numerical analysis,

rather than on their physical significance (as developed for example, in (5)-(8) of Sec. 8.3, (6)-(9) of Sec. 8.3).

Some relatively recent works based on or relatable to Schuster's theory are contained in the papers of Benford [18], [19], [20], Whitney [316], Hulbert [114], Kubelka [145], Middleton [178], a report by Sliepcevitch and associates [287], and a paper by Kottler [142]. A fairly exhaustive bibliography of the two-flow theory may be compiled from the references of the above papers.

The developments of this chapter are drawn in the main from [221], in which the two-flow equations (19) of Sec. 8.3 were first rigorously derived from the equation of transfer and with particular emphasis on the structure of the functions $f(z,\pm)$, $b(z,\pm)$, $a(z,\pm)$, $\alpha(z,\pm)$, $s(z,\pm)$. Reference [221] is also the source of the two-D model and related concepts. The invariant imbedding relation and the principles of invariance for irradiance were developed in [243], along with a Green's function construction of the R and T factors. The latter construction is simply a special case of the method of the interaction principle, and reproduces in miniature the constructions of Chapter XIV of Ref. [251]. Sec. 8.8 is based in the main on [227]. The divergence law of the light field (15) of Sec. 8.8, along with its most general form is given in [220]. A theory of the irradiance vector $\mathbf{H}(x)$, and the corresponding divergence law in vacua is developed in [187] and [188]. Gershun [98] was first to explicitly recognize the importance of the photometric counterpart to the irradiance vector $\mathbf{H}(x)$, and Milne [180] noted the divergence law's occurrence in astrophysical optics. This same law may be found in Chandrasekhar [43]. The canonical form for irradiance, as given in (40) of Sec. 8.9, was developed in [223].

The theory of internal sources given in Sec. 8.5 has been considerably extended in subsequent invariant imbedding studies of linear hydrodynamics.[1-2] These new techniques (derived in the hydrodynamic context) are directly applicable to the Schuster two flow equations with source terms.

The determination of the four transfer functions $R(x,z)$, $T(x,z)$, $R(z,x)$, $T(z,x)$ (cf. (39)-(42) of Sec. 8.7) can be made *simultaneously* via integration of so called *Riccati quartets* of differential equations, as developed recently in linear hydrodynamics.[3]

1. Preisendorfer, R. W., *Forcing Long Surface Waves Through Two-Port Basins. I. Circuit Variables*, NOAA-JTRE-156, HIG-76-11 Hawaii Institute of Geophysics, November 1976.

2. Preisendorfer, R. W., *Forcing Long Surface Waves Through Two-Port Basins. II. Two-Flow Variables*, NOAA-JTRE-157, HIG-76-12 Hawaii Institute of Geophysics, November 1976.

3. Preisendorfer, R. W., *Multimode Long Surface Waves in Two-Port Basins*, NOAA-JTRE-125, HIG-75-4 Hawaii Institute of Geophysics, January 1975.

PART III

THEORY OF OPTICAL PROPERTIES

CHAPTER 9

GENERAL THEORY OF OPTICAL PROPERTIES

9.0 Introduction

This chapter opens Part III of the present work wherein we shall be concerned with the theoretical study of the main optical properties of natural optical media, with particular emphasis on the properties of natural hydrosols such as seas, harbors, and lakes principally under natural lighting conditions. Some of the optical properties to be considered have been introduced as a matter of course in the earlier developments of Part II. Other properties will be defined and studied here for the first time. For example, we have already encountered the volume attenuation function α (Sec. 3.11), the volume scattering function σ (Sec. 3.14), the volume absorption function a (Sec. 4.2). The functions α and σ constitute the inherent local optical properties of radiative transfer theory which are *fundamental* in the sense that they are sufficiently complete to allow the construction, in principle, of all other optical properties of radiative transfer theory. They, however, are not unique in this property: There are other collections of inherent optical properties which are *fundamental*, i.e., which allow a similar construction of the class of optical properties used in radiative transfer theory. It is the purpose of this chapter to define and classify the optical properties normally encountered in radiative transfer discussions, to derive and display some of the manifold interconnections existing among them, and to list some of the fundamental sets of optical properties (such as α , σ mentioned above).

We shall begin in the following section with some general definitions which will help to initially classify the collection of optical properties associated with a given medium into four main groups: the local, global, inherent, and apparent properties. Since the settings of hydrologic optics are most naturally plane-parallel media, we shall for the most part develop and illustrate the optical properties and their interconnections in such media. Thus in Sec. 9.2 the most important apparent optical properties of hydrologic optics will be introduced and some general patterns of geometric and radiometric behavior of these properties will be derived in Secs. 9.3 to 9.5. Section 9.6 collects together for reference the principal optical properties of radiative transfer theory in plane-parallel media.

9.1 Basic Definitions for Optical Properties

The optical properties of a natural optical medium, such as the sea, or a lake, or a portion of the atmosphere, may be broadly grouped into two classes: those that are *inherent* optical properties and those that are *apparent* optical properties. In simple terms, an inherent optical property of a medium is independent of the various possible lighting conditions that may occur in the medium, whereas an apparent optical property varies with the lighting conditions but usually in such a manner that its other regular properties justify its assignation of the title "optical property." More precisely, and in the terms we have used in the closing remarks of Sec. 3.10, we can make the following:

Definition 1: An optical property P of a subset S of optical medium X (P in the form of a number, function, or operator) such that P is independent of the incident radiance distributions on S will be called an *inherent optical property* of S; otherwise, P is an *apparent optical property* of S.

Here are some examples of inherent optical properties: the volume attenuation function α, as defined in Sec. 3.11; and the beam transmittance function T_r as defined in Sec. 3.10. The independence of these properties from the shape and magnitude of ambient radiance distributions is evident at once from an inspection of their definitions. For example, in the case of T_r, it is seen from (2) and (3) of Sec. 3.10 that T_r is independent of the radiance distributions occurring along the path $\mathcal{P}_r(x,\xi)$ in an optical medium X.

As another example of an inherent optical property of a medium, we have the volume scattering function σ. This important concept is developed in three distinct ways throughout the present work so as to establish it to the satisfaction of both theoretical and practical workers in the field. Its empirical definition, the one used in real light fields to obtain the actual values $\sigma(x;\xi';\xi)$, is presented in Sec. 13.6. The independence of σ from the directional structure of the radiance distributions is at once evident from a study of that definition. The definition of σ as the kernel of the path function operator, as given in Example 1 of Sec. 3.17, is precisely what the theoretical worker can best use, and this definition makes manifest the independence of $\sigma(x;\xi';\xi)$ from radiance distributions $N(x,\xi)$ at x. Finally, an approach to the definition of σ that lies half way between the preceding two approaches and which shares the virtues of each, is given in (5) of Sec. 3.14.

Still other examples of inherent optical properties are the volume total scattering function s and the volume absorption function a defined in (3) and (4) of Sec. 4.2. These are seen to be independent of the ambient radiance distributions because they are derived directly from the inherent optical properties α and σ without the use of any further radiometric concepts.

The examples of inherent optical properties just given are all instances of a type of optical property which is associated with points x of the medium X, or very small

volumes of X. It is useful to occasionally emphasize this fact and we provide the basis for the appropriate terminology via:

Definition 2: An optical property of an optical medium X which has the points of X or the points of some subset S of X in its domain of definition, is a *local optical property*; an optical property of X which has X itself or subsets S of X in its domain of definition, is a *global optical property* of X.

Some examples of global optical properties are the R and T operators (or their kernels) for slabs $X(x,z)$ of a plane-parallel medium $X(a,b)$, as given in Sec. 3.6. Here the subsets $X(x,z)$ of $X(a,b)$ comprise the domain of the operators R and T. The R and T *factors* given in Example 1 of Sec. 3.7 are also examples of global optical properties of $X(a,b)$. The R and T operators are at the same time inherent optical properties of their associated plane-parallel media (e.g., $R(x,z)$ is an inherent global optical property of $X(x,z)$). On the other hand the R and T factors of Example 1 of Sec. 3.7 are apparent global optical properties because they vary with the directional structure of the incident radiance distributions on their respective slabs. Observe how the *shape* of the incident radiance distributions in that example were initially fixed and held fixed throughout the medium, and recall the representations of these R and T factors as given by the two-D theory of irradiance models in Chapter 8.

Examples of local apparent optical properties are found in the two-D irradiance models. For example, $\alpha(z,\pm)$, $f(z,\pm)$, $s(z,\pm)$, $a(z,\pm)$ (in (6)-(8) and (13) and (14) of Sec. 8.3) are local apparent optical properties because they incorporate the distribution functions $D(z,\pm)$ or employ radiance distributions in a way that clearly makes them dependent on the angular structure of the radiance distributions. The functions $\tau(z,\pm)$ and $\rho(z,\pm)$ in (3) and (4) of Sec. 8.2 are also local apparent optical properties.

The operators $\rho(y)$ and $\tau(y)$ (or their anisotropic generalizations $\rho_\pm(y)$, $\tau_\pm(y)$) in (3) and (4) of Sec. 7.1 are examples of local inherent optical properties which are distinct from α and σ but which are just as capable as α and σ in developing radiative transfer theory. On the other hand the standard scattering function $S(X;\cdot,\cdot;\cdot,\cdot)$ as given in (1) of Sec. 3.8 is an inherent global optical property which can be used to reconstruct radiative transfer theory from scratch, as is briefly indicated in that section and as is shown in detail in Sec. 126 of Ref. [251]. These observations bring out the idea of an optical property or a set of optical properties (most likely inherent) which can be used to construct all of radiative transfer theory. Since the equation of transfer is a central working tool of the theory, we shall use it as a criterion for deciding whether a set of optical properties is fundamental, and since α and σ form the optical heart of the equation, we shall agree to the following:

Definition 3: A set $\mathcal{P}$ of optical properties is said to be *fundamental* for a medium X with constant index of refraction n if, via $\mathcal{P}$, along with the laws of geometrical

radiometry and the interaction principle, it is possible to derive the equation of transfer for X; or equivalently, if via $\mathcal{P}$ and the same auxiliary means, the volume attenuation function α and the volume scattering function σ in X can be derived.

Some examples of fundamental sets of local optical properties are (α,σ), (α,p) where p is the phase function (see (3) of Sec. 7.12). Furthermore, (α,a), $(\rho_\pm(y), \tau_\pm(y))$ are fundamental sets of local optical properties. All of these are inherent optical properties. Some examples of fundamental sets of global optical properties are the operator pairs (R,T) for slabs $X(x,z)$ in a plane-parallel medium $X(a,b)$, or the operator pairs $(\mathcal{R},\mathcal{T})$ for one-parameter subsets $X(x,z)$ of one-parameter optical media $X(a,b)$, and finally the operator pair $(\mathbf{R},\mathbf{T})$ in (5) and (12) of Sec. 3.17. All of these are inherent optical properties. In the slab examples, there are to be two pairs of R and T operators or two pairs of $\mathcal{R}$ and $\mathcal{T}$ operators associated with *every* subslab $X(x,z)$ of $X(a,b)$. Pairs of optical properties such as (a,s), (α,s), are not fundamental since s, being an integral of σ over Ξ has lost too much information in being formed, so that it is generally impossible, except in the simplest of cases, to retrace the steps to σ. The trivial exceptions occur when, e.g., σ is isotropic, or the medium is purely absorbing so that $\sigma = 0$.

Looking back over the preceding definitions and examples we see that the means of telling one type of optical property from another has been made quite clear. There remains, however, the task of classifying the *origins* of the various optical properties, and of clarifying what is meant by the basic terms "optical property" and "optical medium" hitherto used only with their intuitive meanings. For completeness, these definitions are now given.

Definition 4 (for General Radiative Transfer): An *optical medium* is an ordered septuple $(X,N,n,\alpha,\sigma,\hat{\sigma},N_e)$ where X is a subset of euclidean space, N a radiance function on $X \times \Xi$, n the index of refraction function, α, σ are respectively the volume attenuation and scattering functions, $\hat{\sigma}$ the volume transpectral scattering function, and N_e the true emission radiance function. All these functions are nonnegative, real-valued functions.

The only terms used above that are not formally defined in this work are $\hat{\sigma}$ and N_e (see discussion following (2) of Sec. 3.15). These terms are relatively subordinate as far as magnitude of effects is concerned as they occur in geophysical optics and particularly in hydrologic optics, and for that reason are omitted from the present considerations. However, they generally are to be treated on par with the remaining concepts, and the delineation of their salient properties may be found in Sec. 19 of Ref. [251]. In view of this, for all practical purposes of hydrologic and atmospheric optics, one may adopt as the definition of an optical medium, the following:

Definition 5 (for Geophysical Optics): An *optical medium* is an ordered quintuple (X,N,n,α,σ), where X,N,n,α, and σ are as given in the general definition 4 above. For

brevity, the optical medium, i.e., the quintuple, is usually referred to simply as "X".

Finally, we make clear the meaning of the term "optical property" as it is used in radiative transfer theory, by means of the following:

Definition 6: An *optical property* of an optical medium X is any interaction operator s, including its components s_{ij}, if any, arising from the use of the interaction principle in X.

Several comments can be made on the preceding definition which will help tie together the ideas of this and the following sections. First, we deduce from definitions 1 and 6 that the inherent optical properties are precisely those which arise when the incident radiometric quantities used in the interaction principle are *radiance distributions*; all other types of incident radiometric quantities by definition give rise to apparent optical properties. Second, when S in the interaction principle of Chapter 3 is a one, two, or three dimensional subset of X, then the associated optical property is a global property. If S is a subset of X consisting of one point (a zero dimensional subset of X) then the associated property is a local property. Finally, it is helpful to distinguish between optical properties which are associated with diffuse or n-ary (in other words *decomposed*) radiometric functions and undecomposed functions (re: (19) through (22) of Sec. 5.1 and Sec. 8.4). The undecomposed radiometric functions are those that are *directly observed* in nature with the standard radiant flux meters and their variant instrumental forms; and to emphasize this feature, we shall in the present work also refer to such radiometric quantities and their associated optical properties as *directly observable*.

9.2 Directly Observable Quantities for Light Fields in Natural Hydrosols

In this section we introduce a set of directly observable, apparent optical properties which have been found to be of extraordinary power in the radiometric description of natural hydrosols. Because of the importance of these properties we shall introduce them and sketch their logical and historical background in some detail.

Introduction

The increasing accuracy of experimental determinations of the optical absorption and scattering properties of oceans, lakes, and harbors has necessitated exact knowledge of the possible interrelations between these properties. The classical one-D model of the two-flow analysis of the light field has been an important source of such relations (Sec. 8.6). While it continues to be a source of useful simple models for engineering calculations, it has become inadequate to collate the data of basic experimental research

The general two-flow theory of Sec. 8.3 was formulated to meet these new experimental needs. The two-flow formulation is based on the exact equation of transfer for the light field in scattering-absorbing media. The main results accumulated below give useful relations among the magnitudes of the downwelling and upwelling irradiances, their depth rates of change, and the scattering and absorption properties of natural hydrosols. The formulations are applicable in general to arbitrarily stratified plane-parallel media.

To gain an adequate insight into the formulations below, it is necessary to recall some of the historical background of the classical one-D irradiance theory, and hence radiative transfer theory itself. Radiative transfer theory arose, in part, in the attempts of astrophysicists to understand the problem of the passage of light through stellar and planetary atmospheres. The theory subsequently was found to be of use in terrestrial settings and consequently was directed by geophysicists to the problem of the passage of light through natural optical media such as the atmosphere of the earth, and its waters such as oceans and lakes. While both fields of astrophysics and geophysics share a common interest in the theory of optical media that scatter and absorb light, their experimental procedures are vastly different: The geophysicist has easy access to his optical media and can amass first hand data from even the remotest places of a natural aerosol or hydrosol. This fact has far-reaching effects on the form of the attendant theory of radiative transfer. The astrophysicist has no such direct access and is forced to devise radically different mathematical models, and limit his explorations to theoretical studies of such models and their observable consequences.

Thus when the geophysicists adopted the more useful of these astrophysical models, they inherited, for better or worse, certain simplifying assumptions built into them, assumptions which had been explicitly and deliberately introduced to make them analytically tractable and of some practical worth. In the course of time, as the study of the light fields in natural aerosols and especially natural hydrosols became progressively more exact and exhaustive, these adopted models, which were extremely valuable in planning and interpreting the early geophysical experiments, became progressively inadequate, especially in their predictive and correlative powers. However, the formulas connected with these models had proved so convenient, and their simplicity so appealing, that their impending loss set off a search for more exact replacements which were to retain, whenever possible, the utility and simplicity of the classical formulas. The results of such a search are summarized in the discussion that follows. The model considered is the modern version of the two-flow theory as developed in Sec. 8.3. In what follows we will use the general formulation of the two-flow equations, and require precise knowledge of the assumptions adopted in the classical formulations. The requisite groundwork has been covered in adequate detail in Chapter 8, and the following discussion will draw freely from the results of that chapter.

While the motivation and main emphasis of this study is in connection with that branch of radiative transfer known as hydrologic optics, it would perhaps be of interest to observe that the results summarized below can actually be applied to the experimental study of the transfer of radiation through any arbitrarily stratified plane-parallel medium with arbitrary incident lighting and reflecting boundary conditions.

Classical Two-Flow Theory: The Theoretical K-Functions

In the present discussion we will prepare the way for the desired general formulations by returning to the classical one-D model of the two-flow theory, as it has been developed and used in hydrologic optics, and isolating those formulas which have been of importance in checking and collating experimental data. The key notion in all that follows is that of the diffuse absorption coefficient k and its generalizations.

As we saw in (36) of Sec. 8.6, the classical one-D theory for decomposed irradiance fields in natural hydrosols is embodied in the pair of equations,

$$\begin{aligned} \frac{dH^*(z,-)}{dz} &= -\ [a^* + b^*]\ H^*(z,-) + b^*H^*(z,+) + f^oH^o(z,-) \\ -\ \frac{dH^*(z,+)}{dz} &= -\ [a^* + b^*]\ H^*(z,+) + b^*H^*(z,-) + b^oH^o(z,-) \end{aligned} \qquad (1)$$

Here $H^*(z,-)$ and $H^*(z,+)$ are the irradiances at depth z induced, respectively, by downwelling diffuse flux (-) and upwelling diffuse flux (+). The term, "diffuse flux," as defined in (22) of Sec. 5.1, refers to flux which has been scattered one or more times. $H^o(z,-)$ is the downwelling residual or reduced irradiance at depth z and refers to flux which has not been scattered. The upward residual irradiance $H^o(z,+)$ is zero; a^*, b^* are the volume absorption and backward scattering coefficients for the diffuse flux; f^o and b^o are the forward and backward scattering coefficients for the downwelling residual flux. The quantities a^*, b^*, f^o, and b^o are assumed constant and known, $H^o(z,-)$ is assumed known for each depth z, and is given explicitly in (22) of Sec. 8.4.

Diffuse Absorption Coefficient k

The parameter which plays the greatest single role in the classical one-D theory is the so-called diffuse absorption coefficient k, given in the form (re: (32) of Sec. 8.6).

$$k = +\ [a^*(a^* + 2b^*)]^{1/2} = 2[a(a + b^*)]^{1/2} \qquad (2)$$

The alternate expression follows from the assumption (usually made in classical applications) that the diffuse flux is described by a uniform radiance distribution in the upper and lower hemispheres; a is the volume absorption

coefficient, and is related to a^*, under the above assumption, by $a^* = 2a$ (re: (27) of Sec. 8.6). From a mathematical point of view, k is a natural consequence of the solution procedure for (1), which yields the general solutions:

$$H^*(z,-) = m_+ g_- e^{kz} + m_- g_+ e^{-kz} - A_-(z) H^\circ(0,-)$$

and (3)

$$H^*(z,+) = m_+ g_+ e^{kz} + m_- g_- e^{-kz} - A_+(z) H^\circ(0,-) \quad .$$

Here:

$$g_- = 1 - \frac{a^*}{k} \quad , \quad g_+ = 1 + \frac{a^*}{k} \quad ,$$

and m_+, m_- are constants of integration determined by the values $H^*(0,-)$, $H^*(z_1,+)$, and $H^\circ(0,-)$ where z_1 is the depth of the medium. Finally, $A_\pm$ is a determinable function (re: (67) of Sec. 8.5) whose specific form is not of interest at present.

In order to fully understand the significance of the experimental K-functions discussed below, we underscore the fact that k, as represented in (2) and as used in the system (3), is a purely mathematical construct: It is devoid of any physical meaning as it emerges from the solution procedure. It has generally no basis in any realizable physical operation. Only in the case of an optically infinitely deep medium irradiated by a uniform radiance distribution may a physical interpretation be assigned to k. For, in this case, the system (3) is such that we may deduce the following representations of the undecomposed irradiances (compare with (22) and (23) of Sec. 8.6):

$$H(z,-) = H^\circ(z,-) + H^*(z,-) = m_- g_+ e^{-kz} = H(0,-) e^{-kz}$$

(4)

$$H(z,+) = H^\circ(z,-) + H^*(z,+) = m_- g_- e^{-kz} = H(0,+) e^{-kz}$$

and

$$H^\circ(z,+) = 0$$

In order to obtain the system (4) directly from (3), it was assumed that $H^\circ(z,-)$, the reduced downwelling irradiance, is associated with an angularly uniform radiance distribution, and behaves with depth in the following manner:

$$H^\circ(z,-) = H^\circ(0,-) e^{-\alpha^\circ z} \quad ,$$

where $\alpha^\circ = \alpha D^\circ$, and $D^\circ = 2$. Furthermore, $f^\circ = f^*$, $b^\circ = b^*$, $a^\circ = a^*$. Under this assumption, along with $H^\circ(z,+) = 0$, it turns out $A_+(z) = 0$ and that $A_-(z) = e^{-\alpha^\circ z}$.

The final step to (4) was to deduce that $m_+ = 0$, and to recall that the directly observable irradiance $H(z,-)$ is the sum of $H(z,-)$ and $H^*(z,+)$. It is now clear that (4) could also be arrived at using the one-D model for the undecomposed irradiance field (re: (22) and (23) of Sec. 8.6).* In the above case, and only that case, is it possible to interpret K as the decay rate of the irradiance for both streams; and the system (4) suggests the following operational definition for k :

$$k = \frac{-1}{H(z,-)} \frac{dH(z,-)}{dz} = \frac{-1}{H(z,+)} \frac{dH(z,+)}{dz} \quad . \tag{5}$$

Even with this interpretation of k , its formal nature is inescapable; it is a rare medium, indeed, which exhibits a light field of *exactly* the structure described by the system (4).

The term "diffuse absorption coefficient," for k stems from the following three observations: first, and least important is the fact that k is defined in terms of coefficients a^* , b^* which are associated with the diffuse flux making up the light field. Second and more important is the fact that in the extreme case where the medium exhibits only absorption, we have formally, $k = a^*$. And finally, if the medium exhibits no absorption, $k = 0$ (see also (13) of Sec. 10.8).

The R-Infinity Formulas

In addition to the determination of the depth dependence of the values $H(z,-)$ and $H(z,+)$, the experimental determination of the ratio $H(z,+)/H(z,-)$ is of interest because knowledge of this ratio aids in the solution of underwater visibility and photography problems and certain aspects of marine biology problems.

The classical two-flow model yields several expressions for this ratio, the most important being:

$$R_\infty = H(z,+)/H(z,-) = g_-/g_+ = (k - a^*)/(k + a^*) \tag{6}$$

which follows directly from (4), and which was derived earlier in (21) of Sec. 8.6. By using the definition of k , and some simple algebraic rearrangements, (6) may be cast into either of the following two equivalent forms:

$$R_\infty = \frac{a^* + b^* - k}{b^*} \quad , \tag{7}$$

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*This is the more direct of the two methods. However, the alternate route, patterned after the classical approach, is retained to reinforce the point made in this section, namely the inadequacy of the classical approach to form a theory of *directly observable* radiometric concepts in natural optical media. (See appendix, sheet 1 of [210] for original derivation details.)

$$R_\infty = \frac{b^*}{a^* + b^* + k} \quad . \tag{8}$$

The primary virtue of these formulas lies in their simple, explicit inclusion of b^*, whose magnitude may then be estimated if the remaining quantities are known.

Finally, by using the identity

$$\alpha^* = a^* + s^* = a^* + f^* + b^* \quad ,$$

we have:

$$R_\infty = \frac{\alpha^* - f^* - k}{b^*} \quad , \tag{9}$$

and alternately:

$$R_\infty = \frac{b^*}{\alpha^* - f^* + k} \quad . \tag{10}$$

In the above formulas, α^* is the volume attentuation coefficient for the diffuse flux; α^* is related to the basic volume attenuation coefficient α, under the one-D assumptions, by the formula: $\alpha^* = 2\alpha$; α is the sum of the volume absorption coefficient a, and the volume total scattering coefficient s. If α is known, (9) and (10) may then be used to estimate f^*.

The Inequalities

One final set of relations which have consistently proved their usefulness in checking the experimentally obtained values of optical constants of turbid media are the inequalities:

$$a^* \leq k < \alpha \quad . \tag{11}$$

The left inequality follows from (6) by observing that R_∞, k, and a^* can never be negative. It also may be obtained from (2) by omitting the nonnegative number "$2b^*$" from the binomial.

The right-hand inequality is more difficult to establish using simple models. It can be established only under certain conditions on the lighting and inherent optical properties of the medium (see, e.g., (9) of Sec. 6.6). Such conditions will come to light in the discussions of k_∞ in (43) of Sec. 10.5. It is on this basis that the right-hand inequality is understood to hold.* The exact version of (11)

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*In media that exhibit no scattering, there can be no diffuse flux. The definition of k is so rigid, however, that a formal contradiction of the inequality $k < \alpha$ can be obtained by setting $s = 0$. For then $b^* = 0$, and $k = a^* = 2a = 2\alpha$.

that holds in real media will be derived below (cf. (26) and (27)).

Observations on Inadequacies of Classical Theory

The increasing inadequacy of the two-flow model is perhaps most succinctly summarized in the preceding discussion of the basic inequalities of the theory. But this is a mathematical matter which should carry less weight than some more striking inadequacies on the experimental front: the observed nonconstancy of R_∞, the lack of an operational definition of k, except in extreme hypothetical cases, and the fact that the basic system (1) refers to nonobservable quantities, namely $H^*(z,\pm)$. However, most of the trouble stems from the fact that the angular structure of *both* the diffuse and reduced radiance distribution have, for all depths, been assigned a fixed uniform structure in both the upper and lower hemispheres. From this follows the constancy of all the absorption and scattering coefficient functions, and that of R_∞, which is in direct conflict with experimental observations.

The question then arises: Is there some procedure which retains the conceptual simplicity of the classical two-flow analysis and at the same time incorporates an accurate representation of the effects of the depth dependence of the observable radiance distributions? The answer is yes. The details of the solution are presented below.

Exact Two-Flow Theory: Experimental K Functions and R Functions

We turn now to the systematic development of the directly observable counterparts to the concepts occurring in the one-D and two-D models for irradiance fields. First of all, the distribution functions for the downwelling and upwelling streams are defined by writing:

$$\text{"}D(z,-)\text{"} \quad \text{for} \quad \frac{h(z,-)}{H(z,-)} \tag{12}$$

and

$$\text{"}D(z,+)\text{"} \quad \text{for} \quad \frac{h(z,+)}{H(z,+)} \tag{13}$$

The quantities $h(z,-)$ and $h(z,+)$ are the scalar irradiance induced, respectively, by the down and upwelling streams at depth z (cf. (11) of Sec. 2.7). They may be measured by suitably shielded spherical collectors, whereas, the ordinary irradiances $H(z,-)$ and $H(z,+)$ are measured by flat plate collectors. The ratio of a hemispherical scalar irradiance to an irradiance provides a simple means of characterizing the directional distribution of flux in each stream, as we saw in some detail in Sec. 8.5. But their usefulness extends considerably beyond this service, as we shall see below.

In what follows, $H(z,-)$ and $H(z,+)$ are the undecomposed downwelling and upwelling irradiances: the irradiances actually measured by horizontal flat plate collectors. The exact equations governing these source-free media are given by (19) of Sec. 8.3 and are:

$$\boxed{\begin{aligned} \frac{dH(z,-)}{dz} &= -[a(z,-)+b(z,-)]H(z,-)+b(z,+)H(z,+) \\ -\frac{dH(z,+)}{dz} &= -[a(z,+)+b(z,+)]H(z,+)+b(z,-)H(z,-) \end{aligned}} \tag{14}$$

The other functions appearing in (14) are the absorption, $a(z,\pm)$, and backward scattering, $b(z,\pm)$ functions for the respective streams; their exact definitions and properties are given in (8) and (13) of Sec. 8.3. For brevity, source terms have been omitted from the system (14); that is, terms which describe self-luminous or transpectral sources of radiant flux distributed throughout the medium. Their inclusion does not modify the essential forms of the following results. However, as noted in Sec. 1.2 and in the discussion of definitions 4 and 5 in Sec. 9.1, true sources are relatively scarce in natural hydrosols, and so may be omitted from most basic discussions.

In (5) the diffuse absorption coefficient was expressed as the logarithmic derivative of the irradiances of each flow. It was shown that this simple operational characterization was possible only in a very special set of circumstances in the classical two-flow theory. It turns out, however, that such an operation is the natural way to characterize the decay (and growth) rate of each stream in real media. Each stream in general has its own depth rate of change. These rates of change are not completely independent, as we shall see below, and they are generally different in magnitude. Thus we are led to write:

$$\begin{aligned} &\text{"}K(z,-)\text{"} \quad \text{for} \quad \frac{-1}{H(z,-)}\frac{dH(z,-)}{dz} \;, \\ &\text{"}K(z,+)\text{"} \quad \text{for} \quad \frac{-1}{H(z,+)}\frac{dH(z,+)}{dz} \;. \end{aligned} \tag{15}$$

We call $K(z,\pm)$ the *K-function* (or absorption function) for downward (-) or upward (+) irradiance. Just as the K-functions for each stream are found to change with depth, so does the irradiance ratio, where we write:

$$\text{"}R(z,-)\text{"} \quad \text{for} \quad \frac{H(z,+)}{H(z,-)} \;, \tag{16}$$

and hence its reciprocal also changes with depth z, where we write:

$$\text{"}R(z,+)\text{"} \quad \text{for} \quad \frac{H(z,-)}{H(z,+)} \;. \tag{17}$$

We call $R(z,\pm)$ the *reflectance function* for downward (-) or upward (+) irradiance. The term $R(z,-)$ in (16) is the exact experimental counterpart to R_∞, and may be thought of as the reflectance of the hypothetical plane surface at depth z, with respect to downwelling flux. A similar interpretation exists for (17).

From (14) and the preceding definitions:

$$\boxed{K(z,-) = a(z,-) + b(z,-) - b(z,+)R(z,-)} \quad , \qquad (18)$$

$$\boxed{-K(z,+) = a(z,+) + b(z,+) - b(z,-)R(z,+)} \quad , \qquad (19)$$

which show how the experimental K-functions are linked to the absorption and scattering properties of the medium. From these, we immediately obtain the exact counterparts to (7) and (8):

$$\boxed{R(z,-) = \frac{a(z,-) + b(z,-) - K(z,-)}{b(z,+)}} \qquad (20)$$

$$\boxed{R(z,-) = \frac{b(z,-)}{a(z,+) + b(z,+) + K(z,+)}} \qquad (21)$$

While the overall resemblance to (7) and (8) is quite evident, care should be taken to distinguish the relatively subtle roles now played by the functions associated with each flow.

The counterparts to (9) and (10) are obtained by making use of the general identity ((18) of Sec. 8.3):

$$\alpha(z,\pm) = a(z,\pm) + s(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm) \quad ,$$

Thus, in general:

$$\boxed{R(z,-) = \frac{\alpha(z,-) - f(z,-) - K(z,-)}{b(z,+)}} \qquad (22)$$

$$\boxed{R(z,-) = \frac{b(z,-)}{\alpha(z,+) - f(z,+) + K(z,+)}} \quad . \qquad (23)$$

The Basic Reflectance Relation

The general counterpart to (6) is singled out for special attention because it is the most useful representation of relfectance functions in practice. It relates the six direct observables: the two K-functions, the two distribution functions, the volume absorption function, and the reflectance function. It therefore may replace (6) by providing an exact formula to check the consistency of the experimentally determined values of these functions. Furthermore, from a theoretical point of view, the general counterpart to (6) is closely related to the divergence relation for the light field (re: (18) of Sec. 9.8) in fact directly derivable from it, as shown below. Alternate derivations, of course, can be made directly from the system (14). Now, for the derivation at hand, the divergence relation for the light field in stratified media may be written in the form:

$$\frac{d\bar{H}(z,+)}{dz} = a(z)h(z) \quad , \tag{24}$$

where $\bar{H}(z,+) = H(z,+) - H(z,-)$, and where $h(z)$ is the scalar irradiance at depth z, and $a(z)$ is the value of the volume absorption function at depth z.

Rewriting this with the help of (9) of Sec. 2.7, as:

$$\frac{dH(z,+)}{dz} - \frac{dH(z,-)}{dz} = a(z)[h(z,+) + h(z,-)] \quad ,$$

and dividing each side by, say $H(z,-)$, we have:

$$\frac{H(z,+)}{H(z,-)}\frac{1}{H(z,+)}\frac{dH(z,+)}{dz} - \frac{1}{H(z,-)}\frac{dH(z,-)}{dz} =$$

$$= a(z)\left[\frac{h(z,+)}{H(z,+)}\frac{H(z,+)}{H(z,-)} + \frac{h(z,-)}{H(z,-)}\right]$$

Applying the appropriate definitions, this may be rewritten:

$$- R(z,-)K(z,-) + K(z,-) = a(z)[D(z,+)R(z,-) + D(z,-)]$$

$$= a(z,+)R(z,-) + a(z,-) \quad .$$

Solving for $R(z,-)$:

$$\boxed{R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)}} \tag{25}$$

which is the desired exact experimental counterpart to (6).

The Exact Inequalities

To obtain the exact counterparts to the classical inequalities (11), we encounter a reversal of difficulty: the counterpart to $k < \alpha$ is relatively simple to establish, and its validity is completely general; the counterpart to $a^* \le k$ requires additional assumptions, but of a kind which are a consequence of the generality of the present formulations rather than their shortcomings.

The first member of (14) may be written as:

$$\frac{dH(z,-)}{dz} = -\alpha(z,-)H(z,-) + \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi)$$

so that:

$$K(z,-) = \alpha(z,-) - \frac{1}{H(z,-)} \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi) \quad .$$

This representation may be obtained by using the definition of $K(z,-)$, recalling (18) of Sec. 8.3, and the derivation leading to (9) of Sec. 8.3. Since the subtracted member of the right side is never negative, we have immediately:

$$K(z,-) \le \alpha(z,-)$$

for all z. Finally, whenever $0 \le K(z,+)$, we have (see note below) from (25):

$$a(z,-) \le K(z,-) \quad ,$$

which establishes the desired inequalities:

$$\boxed{a(z,-) \le K(z,-) \le \alpha(z,-)} \quad , \tag{26}$$

or equivalently:

$$a(z) \le \frac{K(z,-)}{D(z,-)} \le \alpha(z) \quad .$$

and alternately

$$0 \le \frac{K(z,-)}{D(z,-)} - a(z) \le s(z)$$

A corresponding set of inequalities for the upwelling stream may be obtained in a similar way:

$$\boxed{a(z,+) \le -K(z,+) \le \alpha(z,+)} \tag{27}$$

or equivalently:

$$a(z) \leq - \frac{K(z,+)}{D(z,+)} \leq \alpha(z) \quad .$$

and alternately:

$$0 \leq - \frac{K(z,+)}{D(z,+)} - a(z) \leq s(z)$$

The right-hand side of (27), as that of (26), holds in general; the left side of (27) holds whenever $K(z,-) \leq 0$, a condition completely symmetric to the condition, $0 \leq K(z,+)$ used to establish the left side of (26).

The Significance of the Condition $0 \leq K(z,+)$

The significance of the condition, $0 \leq K(z,+)$, used to establish the left side of (26), is quite important; and the adoption of this condition raises some interesting questions. First of all, we observe that if the condition holds, then the denominator of (25) is positive. Since $R(z,-)$ is positive, this then requires the numerator of (25) to be positive, from which the desired inequality follows. But one may ask: is this condition ever violated? In other words, can we ever have: $K(z,+) < 0$? Before answering this, we recall that $K(z,+) < 0$ means that $K(z,+)$ is negative, and physically this means that the function $H(\cdot,+)$ is *increasing* with *increasing* depth at z. A similar interpretation exists for the condition $K(z,-) < 0$. The preceding question may then be put very concretely as follows: As one measures upwelling and downwelling irradiances in a real stratified optical medium, is it ever possible to observe an *increase* in these irradiances as depth is increased? The answer is yes. There are two possible mechanisms which generally allow such a phenomenon to be observed.

The first mechanism is that associated with self-luminous sources within the medium. For example, light-giving organisms distributed in a horizontal layer of water clearly make it possible for increases of irradiance to be observed as the irradiance collectors approach the layer and pass downward through the layer. These layers can occur at quite large depths. Other examples are given by various physical emission processes, fluorescence, i.e., e.g., scattering with change in wavelength, et cetera. The latter mechanisms, as noted several times, ordinarily play a subordinate role in many natural hydrosols,* but in the atmosphere, they can be important. If the emission terms $h_\eta(z,\pm)$ are included in the two-flow equations, this phenomenon can be represented

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*When biological processes in natural hydrosols are of interest in hydrologic optics studies, fluorescence can play import roles in the associated radiative transfer process. In this case transpectral scattering theory (cf. Sec. 19 of [251] is the appropriate theory to use.

explicitly, and a suitable parallel theory can be built up around such a phenomenon.

The second mechanism is that of simple scattering processes: the redirection of radiant flux without change in wavelength. This results in an effective storage of radiant energy within scattering media. It seems plausible that if an increase of irradiance is induced by this mechanism in natural water, the increase should noticeably occur at depths near the surface of the medium. For in these regions of small depth the diffuse light field is still building up in magnitude, and just the right kind of inhomogeneities may possibly contribute to the effect. Generally, in optically deep scattering media, the downwelling diffuse radiant flux is zero at the surface, increases with increasing depth, reaches a maximum at some depth, and then falls off in a more or less exponential way ever afterward. But the diffuse irradiance $H^*(z,-)$ is not directly observed. Superimposed on it is the reduced irradiance $H^o(z,-)$, which clearly must decrease continuously with depth, starting right from the upper surface. Thus, whether or not the directly observable irradiance $H(z,\pm) = H^o(z,\pm) + H^*(z,\pm)$ exhibits any increase with increasing depth clearly depends upon the magnitudes and relative rates of change of each of its components.

A theoretical discussion of the conditions which govern the growth of the light field in stratified media is out of place here. We merely note in passing that many such conditions can be extracted from expressions like (14), (18), or (19) given above; furthermore, various approximate models of the light field such as the two-D theory discussed in Chapter 8 give very explicit, if only approximate, criteria for the growth of the light field. Some of these possibilities will be explored in Chapter 10. Finally, the possibility of the growth of *radiance* values has been predicted by a simple model for radiance distributions in natural hydrosols. This prediction has been verified by experiment. The model is based on the canonical form of the equation of transfer; see in particular (12) of Sec. 4.5.

Relative Magnitudes of H and K Functions

For source-free stratified media, we can make several general observations about the relative magnitudes of the observable H-values and K-values. These have been of help in checking and collating experimental data. First of all, from the integrated divergence relation (24), we deduce that:

$$\overline{H}(z_2,+) - \overline{H}(z_1,+) = \int_{z_1}^{z_2} a(z)h(z)dz \geq 0 \quad ,$$

so that:

$$\boxed{\bar{H}(z_2,-) \le \bar{H}(z_1,-)} \tag{28}$$

which demonstrates that the *net downward* irradiance function $\bar{H}(\cdot,-)$ $(= H(\cdot,-) - H(\cdot,+))$ never increases with depth. Here z_1 and z_2 are any two depths, z_2 being the greater. If the medium is, in particular, finitely deep with a bottom surface whose reflectance r is such that $0 \le r \le 1$, (re: *discussion of solutions*, (6) and (7) of Sec. 3.1) then (28) immediately implies that, for all depths z,

$$\boxed{R(z,-) = \frac{H(z,+)}{H(z,-)} \le 1} \tag{29}$$

If the medium is optically infinitely deep, we have a similar result. For in this case, we have for all depths z:

$$\bar{H}(z,-) = \int_z^\infty a(z')h(z')dz' \ge 0$$

so that

$$H(z,+) \le H(z,-) \quad ,$$

from which (29) follows once more.

We can derive a correspondingly general inequality that must hold between $K(z,-)$ and $K(z,+)$. From (25):

$$K(z,-) - K(z,+)R(z,-) = a(z,-) + a(z,+)R(z,-) \ge 0 \quad ;$$

whence:

$$K(z,+)R(z,-) \le K(z,-) \tag{30}$$

or equivalently:

$$\boxed{\frac{dH(z,-)}{dz} \le \frac{dH(z,+)}{dz}} \quad .$$

This relation throws some light on the question raised above. Relation (30) shows that if $K(z,-)$ is negative, then necessarily $K(z,+)$ is negative, too. Conversely, if $K(z,+)$ is positive, then so must $K(z,-)$ be positive. Finally, (30) hints at real situations in which $K(z,+)$ may well be negative while $K(z,-)$ is positive. Ideal examples of each of these three situations are easily found; however, occurrences in real media have not yet been sought.

Characteristic Equation for $K(z,\pm)$

The classical two-flow theory gives a convenient expression for k in terms of absorption and scattering coefficients as in (2). There is a remarkable corresponding formula which characterizes $K(z,-)$ and $K(z,+)$ in addition to (18) and (19). This exact counterpart to (2) is obtained by eliminating $R(z,-)$ from (20) and (21). The result is:

$$\boxed{1 = \frac{b(z,-)}{K(z,-) - a(z,-)} - \frac{b(z,+)}{K(z,+) + a(z,+)}} \tag{31}$$

That this is the general counterpart to (54) of Sec. 8.5 may be verified for example by setting, as such a verification requires, $b(z,-) = b(z,+) = b^*$, $a(z,-) = a(z,+) = a^*$, and $K(z,-) = K(z,+) = k$. When this is done, (31) reduces to (2).

The Depth Rate of Change of $R(z,-)$

Since the experimental counterpart to R_∞ generally varies with depth, it is of interest to characterize the variation in terms of the experimental K-functions. The desired formula follows immediately from the definition (16) of $R(z,-)$:

$$\boxed{\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)]} \quad . \tag{32}$$

From this we see that the constancy of $R(\cdot,-)$ is equivalent to the equality of $K(\cdot,-)$ and $K(\cdot,+)$. In other words, $R(\cdot,-)$ is constant over any interval (z_1,z_2) of depths when and only when $K(z,-) = K(z,+)$ for every depth z in the interval (z_1,z_2).

Connections Among the K Functions

In this paragraph we will briefly discuss the connection between the k of the classical theory (as in (2)) and the exact K-functions introduced in (5) above. First of all we observe the simple connection that exists between the scalar irradiances $h(z,\pm)$ and the irradiances $H(z,\pm)$ within the framework of the classical theory. Recall that both the diffuse and reduced radiance distributions in both the upper and lower hemispheres are assumed uniform; therefore, for all depths z,

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} = 2 \quad .$$

This means that $h(z,\pm)$ and $H(z,\pm)$ differ multiplicatively only by a fixed factor 2. Thus the operational definition (5) for k lets us conclude:

$$k = - \frac{1}{H(z,\pm)} \frac{dH(z,\pm)}{dz} = - \frac{1}{h(z,\pm)} \frac{dh(z,\pm)}{dz}$$

In other words, the classical theory says that k may be estimated equally well from measurements of scalar irradiances or ordinary irradiances. As demonstrated above (see (12) and (13)) distinctions between h and H are often necessary not only in theory but in careful experimental practice. Consequently, when it becomes necessary to discuss the growth or decay of, say, $h(\cdot,-)$ with depth, its logarithmic derivative is generally considered distinct from that of $H(\cdot,-)$. A similar statement is true for $h(\cdot,+)$. Thus we are led to consider operations of the kind:

$$- \frac{1}{h(z,\pm)} \frac{dh(z,\pm)}{dz} ,$$

and to distinguish these from the operations

$$- \frac{1}{H(z,\pm)} \cdot \frac{dH(z,\pm)}{dz}$$

we write:

$$\text{"}k(z,\pm)\text{"} \quad \text{for} \quad - \frac{1}{h(z,\pm)} \frac{dh(z,\pm)}{dz} \tag{33}$$

As a mnemonic, we observe that in the exact theory for real media, the little k's go with little h's and big K's go with big H's. In real media the connection between these is:

$$\boxed{k(z,\pm) = K(z,\pm) - \frac{1}{D(z,\pm)} \frac{dD(z,\pm)}{dz}} \tag{34}$$

We may now state the connection we set out to establish: $k(\cdot,\pm)$ and $K(\cdot,\pm)$ are equal over some interval (z_1,z_2) of depths when and only when the distributon function $D(\cdot,\pm)$ is constant over that interval. This is precisely the situation that holds for optically infinitely deep media in the classical theory so that we have in that setting: $k(\cdot,\pm) = K(\cdot,\pm)$. Furthermore, in this case, $R(\cdot,-) = R_\infty$, and thus is independent of depth. From (32) we may then conclude that $K(\cdot,-) = K(\cdot,+)$. The net conclusion is that for each depth z, the four quantities: $k(z,+)$, $k(z,-)$, $K(z,+)$, $K(z,-)$, which are generally four distinct quantities in real media, are constrained in the one-D model of the two-flow theory to be identical, their common value being k, as given by (2). In this way we justify the generally interchangeable use of k and K in any discussion which has the constancy of the distribution functions in the background.

K-Function for Radiance

The K-functions discussed throughout this section are associated with an *exact* formulation of the two-flow analysis of the light field. They are the little k's for the little h's, and the big K's for the big H's. In more detailed experimental studies of the light field, namely those that document the radiance distribution values $N(z,\theta,\phi)$ at each depth z in all directions (θ,ϕ), a corresponding K-function has been found extremely useful in theoretical work (re: (20) of Sec. 4.5 and Secs. 10.5 and 10.6) and in graphical and tabular representations of these distributions. It is defined by writing:

$$"K(z,\theta,\phi)" \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} \tag{35}$$

No confusion should arise from the continued use of the letter "K": (35) will always explicitly exhibit three variables or places for them when clarity is threatened, the other K's only one. We note in passing that this function, analogously to the other K-functions discussed above, has several interesting theoretical consequences in addition to its immediate experimental uses. However, a discussion of these matters is deferred until Chapter 10.

General K Functions

To round out the discussion of the experimental K-functions, we note that all of the K-functions defined above fall into a specific class, each member of which is defined by an operation of the kind:

$$-\frac{1}{A}\frac{dA}{dz} \quad ,$$

where A could be any of the functions: $H(\cdot,\pm)$, $h(\cdot,\pm)$, $\underline{N}(\cdot,\theta,\phi)$. Some further possibilities for A are, $h(\cdot)$, $\overline{H}(\cdot,+)$. Furthermore we observe, by means of the divergence relation (21) or more generally by (15) of Sec. 8.8, that the basic volume absorption function may be defined as the result of the operation,

$$\frac{1}{h}\frac{d\overline{H}}{dz} \quad ,$$

on the two types of irradiances shown. Finally, the K-function (1) of Sec. 4.5 should be noted. On the basis of these examples, it appears that the most general notion of an experimental attentuation function (i.e., a general K-function) is definable by an operation of the kind,

$$\frac{1}{A}\frac{dB}{dz} \quad , \quad \text{or} \quad \frac{\nabla B}{A} \tag{36}$$

on any two *observable* radiometric quantities A and B.

Integral Representations of the K Functions

The K-function for radiance is basic in the same way that radiance itself is basic; that is, as a fountainhead of representations of the various radiometric concepts. Thus, it is easily shown that:

$$K(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\xi)K(z,\xi)\xi \cdot \mathbf{n}d\Omega(\xi)}{\int_{\Xi_\pm} N(z,\xi)\xi \cdot \mathbf{n}d\Omega(\xi)} \tag{37}$$

$$k(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\xi)K(z,\xi)d\Omega(\xi)}{\int_{\Xi_\pm} N(z,\xi)d\Omega(\xi)} \tag{38}$$

$$k(z) = \frac{\int_{\Xi} N(z,\xi)K(z,\xi)d\Omega(\xi)}{\int_{\Xi} N(z,\xi)d\Omega(\xi)} \tag{39}$$

where $k(z)$ is the negative logarithmic depth derivative of scalar irradiance $h(z)$. Equations (37) through (39) are indicative of the type of integral representations of the various K-functions defined in (36).

Integral Representations of the Irradiance and Radiance Fields

It follows at once from the definitions of the various K-functions introduced above that the directly observable irradiances $H(z,\pm)$ can be given the following integral representations:

$$H(z,\pm) = H(x,\pm) \exp \left\{ - \int_x^z K(y,\pm)\ dy \right\} \tag{40}$$

where x,y,z are three depths in stratified plane-parallel media $X(a,b)$ such that $a \leq x \leq y \leq z \leq b$. Similarly:

$$N(z,\xi) = N(x,\xi) \exp \left\{ - \int_x^z K(y,\xi)\ dy \right\} \tag{41}$$

Since $K(z,\pm)$ and $K(z,\xi)$ are thus observed to play the general roles of absorption functions analogously to a and k, we can alternately refer to them as *absorption functions* for H or N, as the case may be. (See (29) of Sec. 9.3.)

As an example of the use of (40), let us determine the K-function belonging to a spherically symmetric light field about a point source, imbedded in a natural or laboratory hydrosol. The two-flow equations governing radiative transfer across spherical surfaces of radius r and concentric with the source are governed by (46) of Sec. 8.6 in which now $\nabla \cdot \mathbf{H}(z,\pm)$ takes the form:

$$\nabla \cdot \mathbf{H}(z,\pm) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 H(r,\pm))$$

where $H(r,+)$ is the centripetal flux and $H(r,-)$ is the centrifugal flux at radius r. Hence for a steady spherically symmetric light field, the negative logarithmic derivative of $H(r,-)$ with respect to r is:

$$K(r,-) = a(r,-) + b(r,-) - b(r,+)R(r,-) + \frac{2}{r} \quad . \tag{42}$$

A similar formula holds for the centripetal flux by suitably changing signs in the arguments (cf. (19)).

If r and s are any two radii with $r \leq s$, then (40) and (42) yield the formula:

$$\frac{H(s,-)}{H(r,-)} = \left(\frac{r}{s}\right)^2 T_a(r,s) T_b(r,s) \tag{43}$$

where we have written:

$$\text{"}T_a(r,s)\text{"} \quad \text{for} \quad \exp\left\{ - \int_r^s a(u)D(u,-)\,du \right\}$$

$$\text{"}T_b(r,s)\text{"} \quad \text{for} \quad \exp\left\{ - \int_r^s [b(u,-) - b(u,+)R(u,-)]\,du \right\}$$

Here T_a and T_b are special transmittances built up from absorption and backscatter coefficients, respectively. For media with relatively small $b(u,\pm)$ it follows that $T_b = 1$, and so a practical rule of thumb for K in spherically symmetric fields is:

$$K(r,-) \simeq a(r,-) + \frac{2}{r} = a(r)D(r,-) + \frac{2}{r}$$

and (43) becomes

$$H(s,-) = H(r,-)\left(\frac{r}{s}\right)^2 T_a(r,s)$$

$$= H(r,-)\left(\frac{r}{s}\right)^2 \exp\left\{-\int_r^s a(u)D(u,-)du\right\} \quad .$$

A simple, but approximate, operational procedure for measuring volume absorption function in spherically symmetric fields is forthcoming from this. Since

$$\ln\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} = \int_r^s a(u)D(u,-)du \quad ,$$

on holding r fixed and varying only s :

$$a(s) = \frac{1}{D(s,-)} \cdot \frac{d}{ds} \ln\left\{\frac{r^2H(r,-)}{s^2H(s,-)}\right\} \tag{44}$$

where $D(s,-)$ is the value of the distribution function $D(\cdot,-)$ at radial distance s from the source, and $a(s)$ the required value of the volume absorption function at the same radial distance s. This method of determining the volume absorption function supplements those discussed in Sec. 13.8. Finally, the preceding example shows how, with only minor modifications, all the exact two-flow theory formulas for stratified plane-parallel fields can be used to obtain their correspondents in stratified spherical fields--i.e., spherically symmetric fields of irradiance.

9.3 <u>The Covariation of the K Function for Irradiance and Distribution Functions</u>

The purpose of this section is to establish the theorem that at arbitrary fixed depths z the attenuation function value $K(z,-)$ and the distribution function value $D(z,-)$ vary *directly* (but not necessarily linearly) one with the other, in all steady state stratified real plane-parallel media whose volume scattering functions are predominantly forward scattering. In this way we establish a useful criterion for the behavior of $K(z,-)$ in terms of the intuitively simpler concept $D(z,-)$. The theorem is expected to find its greatest use in natural hydrosols. By way of background to these results we now discuss in some detail the physical significance of $K(z,-)$ and $D(z,-)$.

Some Elementary Physical and Geometrical Features of $K(z,-)$ and $D(z,-)$

It is a well-known fact in hydrologic optics that the amount of light in a natural hydrosol such as an ocean or deep lake decreases essentially in an exponential manner with depth from the surface of the hydrosol. The simplest

models of the light field exhibit this fact (cf. (22) of Sec. 8.6). This fact may be expressed succinctly as:

$$H_z = H_o e^{-kz} \tag{1}$$

where H_z is the amount of radiant flux of a given wavelength falling downward on a unit horizontal area (i.e., the *irradiance*) at depth z, and K is a constant determined by measuring the slope of a semilog plot of measured values of H_z versus depth z. The quantity K has dimensions: per unit length, and is usually called the *attentuation coefficient* for irradiance, for the medium under study. Like H_z the quantity K depends implicitly on a specific wavelength λ of the radiant energy penetrating the hydrosol. For many engineering questions, questions of underwater photography, television and visibility, and for many purposes of marine biologist, the story of H_z and K may appropriately end with equation (1). However, for the more demanding purposes of geophysicists charged with the tasks of determining the fundamental optical constants of natural hydrosols, and for those who must make sense out of the experimental data leading to the numerical determination of the fundamental constants, the story of H_z and K has only begun to be told with equation (1).

Confronting these latter investigators in their quest for precisely measured radiometric quantities which must be interrelated by consistent rules of calculation, is a wealth of intricate and nonlinear detail in the depth behavior of H_z. The simple exponential behavior of H_z as summarized in equation (1) must now as a matter of experimental expediency be discarded and in its stead be made to appear a more detailed formula exhibiting the same general outlines of (1), but containing now all the potential variations which may be uncovered in a careful documentation of the light field in real hydrosols. As we saw in (40) of Sec. 9.2, this more detailed formula can take the form:

$$H(z,-) = H(0,-) \exp \left\{ - \int_0^z K(z',-)dz' \right\} \quad . \tag{2}$$

Equation (2) is the physicists' generalization of the mathematician's simple model of the light field expressed in equation (1). Let us examine (2) in detail and thereby uncover its similarities and dissimiliarities with (1). First, $H(z,-)$ represents the *measured* irradiance at depth $z \geq 0$ in the medium produced by downwelling radiant flux on a unit horizontal area. Hence $H(0,-)$ is the downwelling irradiance of such flux on such a surface at depth $z = 0$ measured just below the air-water film. Second, suppose we plot the general equation (2) on semilog paper with depth z as abscissa and $H(z,-)$ as ordinate. Equation (2) gives the value of $H(z,-)$ at a general depth z: Hence equation (2) may then also be used to give the value of $H(z,+\Delta z,-)$, i.e., the downwelling irradiance at a depth $z+\Delta z$, where Δz is any finite positive increment in depth. The appropriate formula for this is:

$$H(z+\Delta z,-) = H(0,-)\ \exp\left\{-\int_0^{z+\Delta z} K(z',-)dz'\right\} .$$

Now, as in elementary calculus, we may approximate the slope of the semilog plot of $H(z,-)$ at depth z by letting z grow by an increment Δz, by finding the new value $H(z+\Delta z,-)$ and then by performing the operation:

$$\frac{\ell nH(z+\Delta z,-) - \ell nH(z,-)}{\Delta z} \simeq \begin{bmatrix}\text{slope of } \ell nH(z,-) \\ \text{at depth } z\end{bmatrix} \tag{3}$$

The smaller the magnitude of Δz, the more accurate is the estimate of the slope of the curve by this operation. Let us now perform the operation on the values of $H(z,-)$ and $H(z+\Delta z,-)$, as given by (1) and (2). From (1):

$$\ell nH(z,-) = \ell nH(0,-) - \int_0^z K(z',-)dz' \tag{4}$$

and from (2):

$$\ell nH(z+\Delta z,-) = \ell nH(0,-) - \int_0^{z+\Delta z} K(z',-)dz' . \tag{5}$$

Inserting these values in (3) we have:

$$\begin{bmatrix}\text{slope of } \ell nH(z,-) \\ \text{at depth } z\end{bmatrix} \simeq \frac{\int_0^z K(z',-)dz' - \int_0^{z+\Delta z} K(z',-)dz'}{\Delta z} . \tag{6}$$

We now recall two elementary facts from integral calculus, the first is:

$$\int_0^{a+\Delta} f\,dz = \int_0^a f\,dz + \int_a^{a+\Delta} d\,dz , \tag{7}$$

which shows how the range $(0,a+\Delta)$ of integration may be broken into two parts: $(0,a)$ and $(a,a+\Delta)$ for any function integrable over $(0,a+\Delta)$; and the second fact that:

$$\int_a^{a+\Delta} f\,dz \simeq \Delta f(a) , \tag{8}$$

whenever Δ is small and f is continuous at a. Applying these two facts to (6), where $K(\cdot,-)$ now takes the place of f in (7) and (8), we have, on application of (7) to (6):

$$\begin{bmatrix}\text{slope of } \ln H(z,-) \\ \text{at depth } z\end{bmatrix} \simeq \frac{\int_z^{z+\Delta z} K(z',-)\,dz'}{\Delta z} \tag{9}$$

Then, applying fact (8) to (9) and letting $\Delta z \to 0$ we have:

$$- \begin{bmatrix}\text{slope of } \ln H(z,-) \\ \text{at depth } z\end{bmatrix} = K(z,-) \quad . \tag{10}$$

Equation (10) tells us that $K(z,-)$ is the negative of the slope of the semilog plot of $H(z,-)$ versus depth z.

Another way of obtaining (10) is to directly differentiate (2), the result being:

$$\frac{dH(z,-)}{dz} = - K(z,-)H(z,-) \quad ,$$

and then solve for $K(z,-)$:

$$K(z,-) = - \frac{1}{H(z,-)} \frac{dH(z,-)}{dz}$$

$$= - \frac{d \ln H(z,-)}{dz} \quad . \tag{11}$$

This latter method is more elegant than the preceding pedestrian method, and thereby brings out more succinctly the geometric meaning of $K(z,-)$.

We may perform the same operation on H_z as given in equation (1). Hence, either by the z - method or the simple derivative scheme shown above, (1) yields:

$$K = - \begin{bmatrix}\text{slope of } \ln(h_z) \\ \text{at depth } z\end{bmatrix} = - \frac{1}{H_z} \frac{dH_z}{dz} \quad . \tag{12}$$

Thus both K and $K(z,-)$ are the negative slopes of the semilog plots of H_z versus depth z. In this way they are similar. But the point where they differ is in the fact that K is independent of depth and that $K(\cdot,-)$ is not independent of depth. And in this difference lies precisely the difference between mathematical fiction and physical reality. Is there a simple explanation for this gap in terms of the accumulated concepts of hydrologic optics? We now consider in detail an explanation of this difference in terms of the currently accepted concepts of radiative transfer theory, as applied to hydrologic optics.

A careful examination of the experimental evidence leading to H_z determinations shows that there are actually two mechanisms in source-free media which may give rise to the gap between the simple classical K and the modern $K(z,-)$. We have up to this point slanted the discussion to bring out only one of these mechanisms, the one which we may term the *physical* (or dynamical) mechanism of the variation of $K(\cdot,-)$. Thus, in the preceding discussion we centered

attention on the *depth behavior* of $H(z,-)$, a behavior which is discerned only as an irradiance probe moves downward into a natural hydrosol and continuously records the magnitude of $H(z,-)$ as z continuously increases. Now if the external lighting conditions on the upper boundary of the source-free medium are arbitrary but fixed in time, that is stationary, and if the hydrosol's fundamental inherent optical properties (namely the volume attenuation α and volume scattering function σ) are arbitrary but stationary, then the depth dependence of $K(\cdot,-)$ is an indicator of the natural interaction of the penetrating photons with the material of the medium; it provides a running account of the depth-rate of attenuation of the number of the downwelling photons streaming past the level z. This dynamical aspect of the variations of $K(\cdot,-)$ will be studied in an elementary but detailed manner in Secs. 10.3 and 10.4. For our present purposes we may thus consider the physical (or dynamical) mechanism behind the depth behavior of $K(\cdot,-)$ to be fairly well understood. There remains, however, the problem of the *geometric mechanism* which also gives rise to variations of $K(\cdot,-)$ to which we now turn, and which forms the central problem under study in the present section.

If, instead of continuously moving the irradiance probe vertically downward in a natural hydrosol, we hold the depth z fixed and let $H(z,-)$ be recorded as a function of time, we would expect in general, because of the continuously changing external lighting conditions above any natural hydrosol, a time dependence of $H(z,-)$ for the fixed depth z. For example, if the probe is set at a depth of five meters at 0600 hours local time in a certain lake, we would expect $H(z,-)$ generally to increase as the sun rises to reach a maximum around 1200 hours, and then to descend at 1800 to a reading comparable to the 0600 reading. On this basically regular diurnal variation of $H(z,-)$ there is superimposed a relatively more rapid variation in $H(z,-)$, induced for example by the movement of clouds or cloud layers between the sun and the hydrosol's surface. Even though the time rates of these latter changes in the external lighting conditions are thousands of times greater than those associated with the more stately diurnal changes, they nevertheless are far too small to cause any true transients in the natural light field. For this reason all these changes in the light field are of a *quasi-stationary* character.

Now, suppose a probe were to be sent quickly downward to accurately record the stationary light field all the while the sun is covered by a cloud and then, when the cloud has just passed away from the sun and allows it to shine with full strength on the hydrosol, another quick but precise probe uncovers the irradiance's depth profile during this sunny condition. If this were done then we would discern upon careful examination of the two semilog plots of $H(z,-)$ that $H(z,-)$ at each given depth on the overcast plot would differ from the value at that same depth on the sunny plot.

Actually, experimenters need not go out of their way to uncover this phenomenon; it crops up with exasperating inevitability in any painstaking (and thus time-consuming)

mapping of the irradiance depth-profile in natural hydrosols. For in a particularly deep hydrosol the determination of the vertical depth-profile may take on the order of a half hour; and when the probe ascends past one of the relatively shallower depths in a check re-run, the sun may have moved as much as five to ten degrees, cloud covers may have changed, thus producing (with most modern equipment) easily measured changes in the structure of the light field in the interim.

When plots of $H(z,-)$ are made of each of these runs, the logarithmic slope $K(z,-)$ on each plot may be noticeably nonconstant with depth; furthermore, and this is the crux of the matter at hand, the $K(z,-)$ value at a fixed depth z for the downward run may differ from that for the upward run. The experimenter would, at this juncture, if he considers this difference in $K(z,-)$ from one plot to another with care, soon realize that the difference may be the result of a superposition of two basically different physical mechanisms: On the one hand there is of course the possibility that the *inherent* optical properties of the medium may have changed during the interval between the times that the probe has visited the given depth z; on the other hand there is the possibility that the external lighting conditions have changed during this period and that this change has in some way become manifest in the difference in the K-values for the given depth z.

If the investigator had made provisions to record during the same period, and over the same depth interval, the radiance distributions within the medium and the *inherent* optical properties of the medium, then he may be able to quantitatively, at least in principle, ascertain, by means of the representations (18) and (19) of Sec. 9.2 and the known structures of $a(z,\pm)$, $b(z,\pm)$, those parts of the differences in the $K(z,-)$ values which are traceable to the changes in the inherent optical properties. Thus, once again we are in possession of sufficient knowledge to understand and cope with the physical aspects of the behavior of $K(z,-)$. There remains, however, that component of the change of $K(z,-)$ at a given depth z which is traceable to the change of the external lighting conditions.

In order to relate the way in which $K(z,-)$ changes with external lighting conditions we must have some means of specifying in a precise manner the concept of "lighting condition." Clearly, in choosing a precise characterization of this concept the absolute amount of the incident radiant flux is of no essential importance. Of critical importance, however, is the *relative* amounts of radiant flux which arrive on the upper boundary of the medium, or on some internal horizontal plane, from the infinite number of possible directions in the hemisphere of incidence. One obvious, and incidentally the most complete, characterization would be by means of the radiance distribution $N(z,\cdot)$ at depth z. While this means may be of considerable use in other contexts, it requires of the experiments a prodigious auxiliary effort to provide the necessary measuring and recording apparatus to obtain this large number of readings. In the interests of experimental expediency, what is needed is a characterization of the relative values of $N(z,\cdot)$ without having to measure

each of the infinitude of values $N(z,\xi)$, $\xi \in \Xi$ where Ξ is the set of all unit vectors in euclidean space. Since relative values are of primary interest, the ideal characterization would then involve not more than two readings of some simple kind.

In searching for two simple radiometric measurements which would be capable of characterizing the relative values of the downwelling radiance distribution, it would be of some convenience to the experimenter if he could make use of his existing data, namely the values of $H(z,-)$. To see what possible choices remain if $H(z,-)$ is adoped as one of the two radiometric measurements which will characterize the *relative* magnitudes of $N(z,\cdot)$, let us express $H(z,-)$ in terms of these values. By definition, we have:

$$H(z,-) = \int_{\Xi_-} N(z,\xi)\xi \cdot \mathbf{n}\, d\,\Omega(\xi) \quad ,$$

where $\mathbf{n}$ is now the unit inward normal to the hydrosol. In practice, $N(z,\cdot)$ is usually determined by a suitably chosen *finite* set $\{ \xi_1, \ldots, \xi_n \}$ of downward directions, and $H(z,-)$ is then computed by the rule (re: (6) through (17) of Sec. 2.5):

$$H(z,-) = \sum_{i=1}^{n} N(z,\xi_i)\xi_i \cdot \mathbf{n}\, \Delta\,\Omega_i \quad ,$$

where $\Delta\,\Omega_i$ is the solid angle associated with the direction ξ_i. The quantities $\Delta\,\Omega_i$ are subject to no specific restrictions except that they be small enough so that $N(z,\cdot)$ is fairly uniform over the associated direction sets and that of course $\Sigma\,\Delta\,\Omega_i = 2\pi$. Now the *relative* magnitudes of the quantitites $N(z,\xi_i)$, $i = 1, \ldots, n$ may be obtained by choosing any one of them, say $N(z,\xi_1)$, and forming the quotients $N(z,\xi_i)/N(z,\xi_1)$, which we shall denote by: "$g(z,\xi_i)$". Then the quotient:

$$\frac{N(z,\xi_1)}{H(z,-)} = \frac{\Delta\,\Omega_1}{\sum_{i=1}^{n} g(z,\xi_i)\xi_i \cdot \mathbf{n}\, \Delta\,\Omega\, i}$$

would serve as a measure of the way in which the $g(z,\xi_i)$ are distributed over the downwelling hemisphere Ξ_-. However, such a measure falls short of being satisfactory for several reasons: First, we have isolated a particular value $N(z,\xi_i)$, and therefore have distinguished it with artificial importance; actually any one of the $n - 1$ other values would serve just as well. Secondly, in order to measure $N(z,\xi_1)$ we would require the services of a specially designed radiance meter, or bring into use for extremely restricted purposes the actual radiance distribution measuring apparatus--which might as well then be used to determine a working sample of $N(z,\cdot)$. Finally, we would prefer to measure an amount of flux comparable in magnitude to $H(z,-)$; for $N(z,\xi_1)$ would generally be a far smaller number than $H(z,-)$

which must then be divided into or divided by $N(z,\xi_1)$, thereby setting the stage for the disruption of numerical accuracy in the data reduction tasks that follow.

The cumulative effect of these observations is to lead to the choice of the sum

$$h(z,-) = \sum_{i=1}^{n} N(z,\xi_i)\Delta\Omega_i \quad ,$$

as the most logical choice for the second radiometric measurement. Its integral representation is:

$$h(z,-) = \int_{\Xi_-} N(z,\xi)\,d\Omega(\xi)$$

In this way we are led to consider the ratio

$$D(z,-) = \frac{h(z,-)}{H(z,-)} \tag{13}$$

which we have encountered before ((15) of Sec. 8.3 and (1) through (7) of Sec. 8.5) and which we have termed the *distribution function.* In terms of the finite-summation representation of $h(z,-)$ and $H(z,-)$, $D(z,-)$ becomes:

$$D(z,-) = \frac{\sum_{i=1}^{n} N(z,\xi_i)\Delta\Omega_i}{\sum_{i=1}^{n} N(z,\xi_i)\xi_i\cdot\mathbf{n}\,\Delta\Omega_i} \tag{14}$$

or, in terms of their integral representations:

$$D(z,-) = \frac{\int_{\Xi_-} N(z,\xi)\,d\Omega}{\int_{\Xi_-} N(z,\xi)\,\xi\cdot\mathbf{n}\,d\Omega} \quad . \tag{15}$$

The quantity $h(z,-)$ may be measured by simple devices, (see [305] and Chapter 13 below). We have seen in Chapter 8 how $D(z,-)$ serves to characterize the distribution of the irradiating flux. Thus, let $n = 1$ in (14), i.e., let the flux come from any single solid angle $\Delta\Omega$ in the general direction ξ; the distribution function then is:

$$D(z,-) = \frac{1}{\xi\cdot\mathbf{n}} \quad . \tag{16}$$

Hence if the irradiation is incident vertically at depth z, $\xi = \mathbf{n}$ and $D(z,-) = 1$. In general, the *more obliquely incident the pencil of radiation,* the smaller the dot product $\xi\cdot\mathbf{n}$, and *the larger is the distribution function.* This

conclusion, although obtained for a special case, is nevertheless valid for any number of pencils.

In what follows we will show in what manner the concepts $K(z,-)$ and $D(z,-)$ are qualitatively interdependent. This will be done by noting the behavior of the intuitively more simple and predictable quantity $D(z,-)$ which, as we have repeatedly seen, serves as a convenient characterization of the lighting conditions. In this way we may gain insight into the dependence of the more complex quantity $K(z,-)$ on given lighting conditions.

The General Law Governing K(z,-) and D(z,-)

We now have sufficient background established so as to derive with some clarity the general law which governs the exact interrelation of $K(z,-)$ and $D(z,-)$. We start with the canonical representation of $K(z,-)$ derived from the equation of transfer in (40) of Sec. 8.9. This representation of $K(z,-)$ now takes the form:

$$K(z,-) = \alpha(z,-) - \frac{\int_{\Xi_-} N_*(z,\xi)\,d\Omega}{H(z,-)} \quad . \tag{17}$$

To obtain (17) from (40) of Sec. 8.9, we set $\Xi_o = \Xi_-$ and set $\mathbf{n} = -\mathbf{k}$ (see Fig. 8.11). An alternate derivation may be obtained by following the steps leading to (26) of Sec. 9.2. To keep the present discussion self contained, we observe that the various terms in (17) are defined as follows:

(i) $\alpha(z,-) = \alpha(z)D(z,-)$, where, of course, $\alpha(z)$ is the value of the volume attenuation function at depth z.

(ii) $N_*(z,\xi)$ is the value of the path function at depth z for the direction ξ. Its analytical representation is:

$$N_*(z,\xi) = \int_{\Xi} N(z,\xi)\sigma(z;\xi';\xi)\,d\Omega(\xi') \quad , \tag{18}$$

where $\sigma(z;\xi';\xi)$ is the value of the volume scattering function at depth z for the incident ξ' and the scattered direction ξ.

In order to extract from (17) the desired explicit connection between $K(z,-)$ and $D(z,-)$, we must reformulate the second term of (17) in such a way as to cause $D(z,-)$ to appear explicitly in that term. Toward this end we first recall that $D(z,-)$ is be definition the ratio of $h(z,-)$ to $H(z,-)$. Now the appearance of the integral in the second term of (17) has all the external earmarks of an $h(z,-)$ type quantity; this is suggested by observing that $N_*(z,\cdot)$ is a radiance function and it is integrated, like $N(z,\cdot)$, over Ξ_-, the set of downward directions. Thus in close

analogy with $h(z,-)$, we write:

$$"h_*(z,-)" \quad \text{for} \quad \int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi) \tag{19}$$

Physically, $h_*(z,-)$ is the downwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth z.

The analogy need not end with (19); in fact we can extend it quite naturally to include the $*$-counterparts to $h(z,+)$, and to scalar irradiance $h(z)$ itself. Thus in analogy to $h(z,-)$ we write (as in (11) of Sec. 2.7):

$$"h(z,+)" \quad \text{for} \quad \int_{\Xi_+} N(z,\xi)\,d\Omega(\xi) \quad . \tag{20}$$

which is the upwelling scalar irradiance at depth z (and which forms the basis for defining $D(z,+) = h(z,+)/H(z,+)$ for the upwelling stream); we further write:

$$"h_*(z,+)" \quad \text{for} \quad \int_{\Xi_+} N_*(z,\xi)\,d\Omega \tag{21}$$

which represents the upwelling scalar irradiance generated by the radiant flux scattered in a unit volume at depth z. Finally, the scalar irradiance $h(z)$ at depth z, being defined by writing (as in (3) of Sec. 2.7):

$$"h(z)" \quad \text{for} \quad \int_{\Xi} N(z,\xi)\,d\Omega \quad , \tag{22}$$

has its scattered analogy in the form $h_*(z)$ where we have written:

$$"h_*(z)" \quad \text{for} \quad \int_{\Xi} N_*(z,\xi)\,d\Omega \quad . \tag{23}$$

Thus corresponding to:

$$h(z) = h(z,-) + h(z,+) \quad , \tag{24}$$

which is based on (9) of Sec. 2.7, we have:

$$h_*(z) = h_*(z,-) + h_*(z,+) \quad . \tag{25}$$

An extremely useful and unexpectedly simple relation subsists between $h(z)$ and its scattered counterpart $h_*(z)$, namely that the ratio of $h_*(z)$ to $h(z)$ is precisely the value of the volume total scattering function s:

$$s(z) = \frac{h_*(z)}{h(z)} \quad . \tag{26}$$

The derivation of this relation along with some suggestions for its use in practical direct determinations of the values of s are given in (7) of Sec. 13.7.

We have now assembled all the required concepts needed for a complete formulation and discussion of the general law governing $K(z,-)$ and $D(z,-)$. Starting with (17) and using definition (19) we may write:

$$K(z,-) = \alpha(z,-) - \frac{h_*(z,-)}{H(z,-)} \quad . \tag{27}$$

By means of the definition of $D(z,-)$ this may be written:

$$K(z,-) = \alpha(z,-) - \frac{h_*(z,-)}{h(z,-)} D(z,-) \quad ,$$

and, finally, using the definition of $\alpha(z,-)$, we may recast this into the presently desired form of the law governing $K(z,-)$ and $D(z,-)$:

$$\boxed{K(z,-) = \left[\alpha(z) - \frac{h_*(z,-)}{h(z,-)}\right] D(z,-)} \tag{28}$$

The Absorption-Like Character of $K(z,-)$

Before presenting the general proof of the covariation of $K(z,-)$ and $D(z,-)$ which will be based on some observations of the structure of (28), we pause to discuss the general radiative transfer nature of the function $K(\cdot,-)$. We will show by general arguments and also by means of a simple example that $K(z,-)$ is essentially an "absorption coefficient," i.e., it serves as an analytical bookkeeping device for the depth rate of *absorption* of the stream of downwelling photons as the stream passes a general depth z.

The heart of equation (28) resides in the difference of the two bracketed terms. The first term is a value of the volume attenuation function which shows that $K(z,-)$ first of all takes cognizance of the simultaneous loss of photons by means of both scattering and absorbing mechanisms. Thus, as a stream of photons crosses an hypothetical surface at depth z, the stream suffers a loss by having some of the photons scattered in all directions about the point of crossing and also by having some of its photons of (implicitly stated) wavelength λ converted into photons of (generally) longer wavelength or into nonradiant energy. Now the second term, involving $h_*(z,-)$ and being *subtracted* from $\alpha(z)$, in effect returns to the downwelling stream all photons which have been scattered in the "forward direction"--the direction of motion of the downwelling stream. The net loss to the downwelling stream is then represented by this difference; it represents the amount of radiant flux per unit area that either has been scattered back into the upwelling stream, or which has suffered true absorption. The first of these alternate possible activities (scattering back into the stream)

points up the dynamic interpretation of the magnitude of $K(z,-)$, and in this respect, it is quite like the local transmittance function $\tau(z,-)$ discussed in connection with (7) of Sec. 8.3. To see the predominantly absorption-like character of $K(z,-)$, we now consider two extreme examples of optical media: One that is purely absorbing, the other purely scattering; we will show that in each of these two extremes the values of $K(\cdot,-)$ tend either immediately or eventually to become directly proportional to the given values $a(z)$ of the volume absorption function. These arguments will be general versions of those leading to the demonstration of the absorption-like character of k in (5) of Sec. 9.2.

First, let $s(z) = 0$ for all z, and let $a(z)$ be arbitrary. We then have the extreme case of a purely absorbing medium. By hypothesis, it follows that $N_*(z,\xi) = 0$ for each z and for all ξ. Furthermore, since $\alpha(z) = a(z) + s(z)$ it follows that $\alpha(z) = a(z)$. With these observations, equation (28) reduces immediately to the simple form:

$$\boxed{K(z,-) = a(z)D(z,-)} \quad . \tag{29}$$

Here $D(z,-)$ generally depends on depth along with $a(z)$. However if depth z and $a(z)$ are held fixed and the external lighting conditions are varied so that $D(z,-)$ is changed, it is quite clear that $K(z,-)$ varies directly and *linearly* with these changes in $D(z,-)$. This is the first and simplest instance of the covariation of $K(z,-)$ and $D(z,-)$. The essentially absorption-like character may be seen by holding $D(z,-)$ fixed. Then $K(z,-)$ varies directly with the value $a(z)$ of the absorption function.

Next, let $a(z) = 0$ for all z in an optically infinitely deep medium in which $s(z) = s > 0$ for all z. We then have the other extreme case of a purely scattering medium. In such a medium, according to (15) of Sec. 8.8, the divergence $\nabla \cdot \mathbf{H}(z)$ of the net-irradiance vector vanishes at each depth z. In particular, in a plane-parallel medium such as that around which the present discussion is centered, this divergence relation takes the simple form:

$$\frac{d\bar{H}(z,-)}{dz} = 0$$

where $\bar{H}(z,-) = H(z,-) - H(z,+)$, $H(z,+)$ being the upwelling irradiance at depth z. It follows that, for every z,

$$\bar{H}(z,-) = c$$

where c is a constant. Since the medium is in steady state and no photons which enter it are ever lost by absorption, we expect that the time rate of emergence $H(0,+)$ of photons per unit are at the surface just equals the time rate of incidence per unit area $H(0,-)$. Hence

$$H(z,-) = H(z,+)$$

for all z. Now the greater the depth z in the medium, the more alike optically are the regions above and below the level z, and the more alike are their optical responses to the equal irradiances impinging on the common internal boundary at depth z. (See, e.g., the discussion of (128)-(131) of Sec. 8.7.) Thus we are led, by considerations of increasing symmetry above and below horizontal planes at great depths, to conclude that $h(z,-) \to h(z,+)$ as $z \to \infty$. Hence at great depths in the present medium, $D(z,-) \to D(z,+)$, so that the angular structure of the upwelling and downwelling radiance distributions become equal (cf. (2) of Sec. 8.5), and in fact (by symmetry) uniform. An immediate consequence of these observations is the fact that:

$$\frac{h_*(z,-)}{h(z,-)} \to \frac{h_*(z,-) + h_*(z,+)}{h(z,-) + h(z,+)} = \frac{h_*(z)}{h(z)} = s$$

in which (26) was used for the last equality. Hence, since equation (28) for the present case of $a(z) = 0$ takes the form:

$$K(z,-) = \left[s - \frac{h_*(z,-)}{h(z,-)} \right] D(z,-) \quad ,$$

we conclude that:

$$K(z,-) \to 0 (= a(z)) \quad , \tag{30}$$

as $z \to \infty$. Thus, in the both extreme types of media, where one is purely absorbing and the other purely scattering, the diffuse attenuation function was shown to tend toward values which are directly proportional to the values of the volume absorption function for these media. It is in this sense that we understand the absorption-like character of $K(z,-)$. Another instance (36) supporting this interpretation of $K(z,-)$ will be encountered as a matter of course in the concluding observations of this section. There we shall present a practical rule of thumb based on the covariation of $K(z,-)$ and $D(z,-)$.

Forward Scattering Media

A necessary prerequisite to the establishing of the general statement of the covariation rule between $K(z,-)$ and $D(z,-)$ is the introduction of the notion of a *forward scattering medium*. Briefly, a forward scattering medium is one for which the volume scattering function has a predominant forward scattering lobe as compared to its backward scattering lobe. We shall assume that the medium is isotropic.

For a precise definition, let $\mathbf{k}$ be the unit outward normal to the plane-parallel medium, and let ξ be an arbitrary element of Ξ; then we write:

$$"s_{-\pm}(z,\xi)" \quad \text{for} \quad \int_{\Xi_\pm} \sigma(z;\xi;\xi')\,d\Omega(\xi') \quad , \tag{31}$$

whenever $\xi \varepsilon \Xi_-$ and:

$$"s_{+\pm}(z;\xi)" \quad \text{for} \quad \int_{\Xi_\pm} \sigma(z;\xi;\xi')d\Omega(\xi') \qquad (32)$$

whenever $\xi \varepsilon \Xi_+$. Some general properties of these s-functions are easily deduced. For example, if the medium is isotropic, then $s_{+-}(z;\xi) = s_{-+}(z;\xi')$, provided that $\xi \cdot \mathbf{k} = -\xi' \cdot \mathbf{k}$. Further, for these ξ, ξ' : $s_{++}(z;\xi) = s_{--}(z'\xi')$. For example, if $\xi \cdot \mathbf{k} = 0$, then $s_{--}(z;\xi) = s(z)/2$ since the axis of the scattering lobe would then lie in the horizontal plane at depth z and the region of integration would be over precisely half the scattering lobe (see Fig. 9.1). Furthermore, for every $\xi \varepsilon \Xi$

$$s_{--}(z;\xi) + s_{-+}(z;-\xi) = s(z) \quad .$$

The connection between $s_{+\pm}(z;\xi)$ and the forward and backward scattering functions for collimated irradiance is quite close and should be noted (see (43) and (44) of Sec. 8.4). The value $s_{--}(z;\xi)$ has the interpretation of a *forward scattering* function for the direction ξ while $\sigma_{-+}(z;\xi)$ has that of a backward scattering function. Physically $s_{--}(z;\xi)$ gives the fraction of flux of a beam of downward direction ξ that is scattered in all downward (forward, with respect to ξ) directions (see Fig. 9.1). If now we write: "Θ" for arc cos $(-\xi \cdot \mathbf{k})$ we may write "$s_{--}(z,\xi)$" as "$s_{--}(z,\Theta)$" and we finally may define a *forward scattering medium as one for which $s_{--}(z,\Theta)$ decreases monotonically with increasing Θ in the range $0 \leq \Theta \leq \pi/2$.* Clearly, (32) implies

$$s_{++}(z,\xi) + s_{+-}(z_1,\xi) = s(z)$$

for all $\xi \varepsilon \Xi_-$, and analogously to $s_{-\pm}(z,\xi)$ we define $s_{+\pm}(z,\Xi)$ as the *forward (+) or backward (-) scattering* function for the upward direction ξ in Ξ_+. Since $s(z)$ is independent of ξ (the medium has been assumed isotropic), we have alternate means of characterizing a forward scattering medium now using $s_{-+}(z,\Theta)$, which of necessity is monotonically increasing with Θ in any forward scattering medium.

The Covariation Rule for K(z,-) and D(z,-)

We may now state the covariation rule:

Let X be an arbitrarily stratified plane-parallel forward scattering medium with given fixed inherent optical properties. If $z \geq 0$ is any fixed depth in X, then $K(z,-)$ and $D(z,-)$ increase, remain constant, or decrease together. Thus if, in particular, over a certain time period $K(z,-)$ and $D(z,-)$ exhibit increments in their values of magnitude $\Delta K(z,-)$ and $\Delta D(z,-)$, then these increments must be simultaneously positive, zero, or negative.

The complete proof of the rule is tedious because it requires an analysis of the total light field throughout the

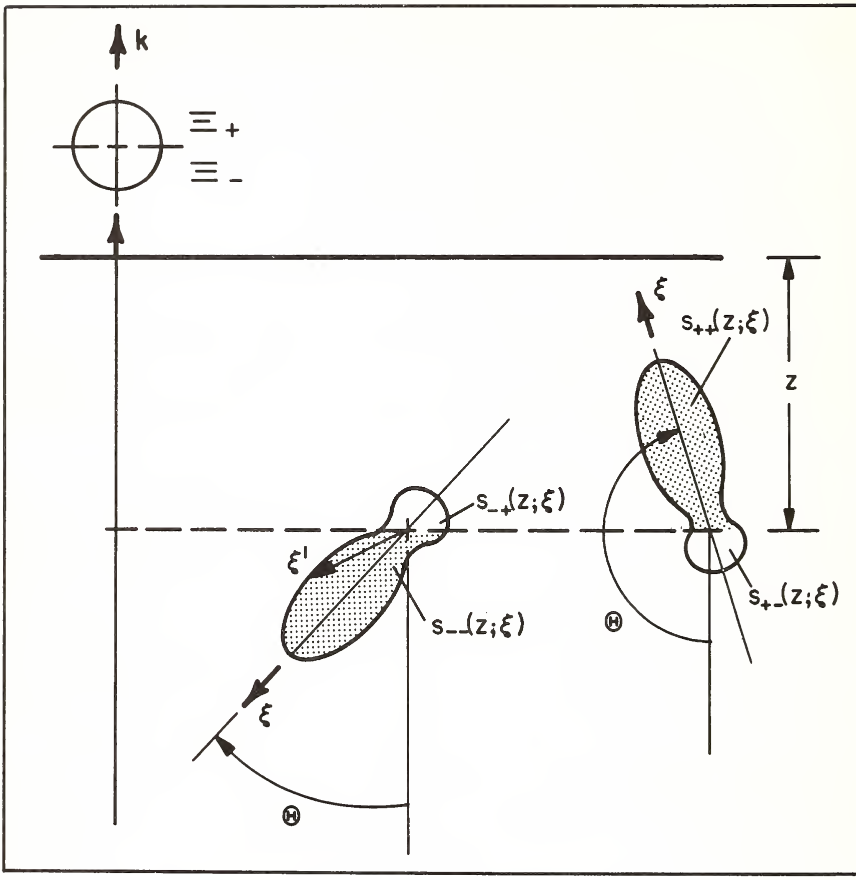


FIG. 9.1 Defining forward and backward scattering functions for the proof of the covariance of $K(z,-)$ and $D(z,-)$.

medium. However, the following argument forms the essential outline of a proof and makes the rule plausible. Suppose that, under the conditions of the hypothesis above, $\Delta D(z,-)$ is positive and $\Delta D(z,+)$ is zero. This implies an increase in the magnitude of $D(z,-)$ which, by our previous discussion of the properties of $D(z,-)$, implies that the preponderance of the radiation has now shifted to make a greater angle with the unit inward normal to an hypothetical surface at depth z. Now the term $h_*(z,-)/h(z,-)$ in (28) is simply a weighted mean of the volume scattering function values $\sigma(z;\cdot;\cdot)$ wherein the weighting factors are the downwelling radiance components. As observed above, the increase in $D(z,-)$ requires that the radiance components carrying the preponderance of the downwelling flux to be now associated with *larger* Θ values in the arguments of forward scattering

function $s_{--}(z,\Theta)$; since the medium is forward scattering, this weighted mean then generally experiences a *decrease* in magnitude; the net result being an *increase* in the bracketed quantity in (28). The total change of $K(z,-)$ is the combination of the increase of the bracketed quantity and the increase in the factor $D(z,-)$; that is, $K(z,-)$ experiences an *increase* in magnitude. Summarizing: An increase in $D(z,-)$ is attended by an increase in $K(z,-)$ all other things remaining fixed in the forward scattering medium. A similar argument may be applied to the assumption that $D(z,-)$ exhibits a decrease. With these two facts established it is then a necessary consequence of continuity in all physical situations that $\Delta K = 0$ whenever $\Delta D = 0$.

Illustrations of the Rule

Example 1. Most natural hydrosols are forward scattering media; in fact $s_{--}(z,\Theta)$, when $\Theta = 0$, occasionally is on the order to ten to twenty times the magnitude of $s_{--}(z,\pi/2)$ over the visible spectrum. The values $\sigma(z;\xi;\xi)$ and $\sigma(z;\xi;\xi_1)$, where $\xi \cdot \xi_1 = 0$, often subtend a ratio of forward to side scattering on the order of 100 to 1, over the visible spectrum. Even more dramatic ratios $\geq$ 100:1 are indicated in Figs. 1.72 and 1.73. The rule may safely be extended even to natural aerosols, these media being predominantly forward scattering; even the borderline case of Rayleigh atmospheres wherein $s_{--}(z,\Theta) = s(z)/2$ for Θ in $[0,\pi/2]$, are subject to the rule, since the square bracketed quantity in (28) does not generally change magnitude, with a change in $D(z,-)$.

Example 2. As a specific illustration, suppose the sky above a lake is completely overcast and that the downwelling distribution and diffuse attenuation functions at some relatively shallow depth z have values D_0 and K_0, respectively, under this overcast condition. Suddenly, the near-zenith sun breaks through the clouds. The resulting value D_1, of the distribution function, is expected to be less than D_0: $D_1 < D_0$, which follows from the fact that the predominant portion of the radiation now comes from generally less oblique directions. It follows that $\Delta D < 0$, that is the increment of $D(z,-)$ is negative, and thus the covariation rule requires that $K(z,-)$ is negative, so that the new value K_1 of the diffuse attentuation function is less than K_0: $K_1 < K_0$.

Example 3. As a final illustration, suppose that we fix attention on a relatively shallow depth in a natural hydrosol which is irradiated by a clear sunny sky for an entire afternoon. As the sun descends, $D(z,-)$ clearly increases because the direction of the predominant portion of the irradiating flux supplied by the sun increases its angle with the vertical (i.e., $1/|\xi \cdot \mathbf{k}|$ increases). The covariation rule would then require that $K(z,-)$ exhibit a corresponding increase.

The Contravariation of K(z,+) and D(z,+)

Up to this point in the discussion we have excluded any detailed mention of the upwelling irradiance $H(z,+)$. However, all that has been discussed for the downwelling stream of radiant energy may be applied *mutatis mutandis* to the upwelling stream, i.e., by replacing minus signs by plus signs in a systematic manner, etc. Therefore, the discussion of the interdependence of $K(z,+)$ and $D(z,+)$ may be conducted relatively quickly by pointing up only the basic differences between the two cases.

Now the first of these main differences between the upwelling and downwelling streams $H(z,+)$ and $H(z,-)$ lies in their magnitudes over the visible portion of the spectrum; the ratio of their magnitudes, $H(z,+)/H(z,-)$ known as the reflectance of the medium at depth z:

$$R(z,-) = \frac{H(z,+)}{H(z,-)}$$

is very nearly and almost universally in the neighborhood of 0.02. Thus $H(z,-)$ is on the order of 50 times the size of $H(z,+)$. Secondly, a fairly constant tie exists between the two streams by virtue of the ratio and sum of their distribution functions. It is found that almost universally over the visible portion of the spectrum (see, e.g., Table 1 of Sec. 8.5).

$$\frac{D(z,+)}{D(z,-)} = 2 \quad , \tag{33}$$

$$D(z,-) + D(z,+) = 4 \quad , \tag{34}$$

expressed to the nearest integer, which then requires that $D(z,-) = 4/3$, and $D(z,+) = 8/3$. The + stream counterpart to (17) is:

$$- K(z,+) = \alpha(z,+) - \frac{\int_{\Xi_+} N_*(z,\xi)\,d\Omega}{H(z,+)} \quad ,$$

which may be reduced to a corresponding expression to (28):

$$\boxed{- K(z,+) = \left[\alpha(z) - \frac{h_*(z,+)}{h(z,+)}\right] D(z,+)} \quad . \tag{35}$$

In much the same manner as $K(z,-)$, the dynamical and geometric mechanisms giving rise to the depth and temporal changes of $K(z,+)$ may be discussed in complete detail. The only precautionary observation that should be made here is that the dynamical mechanisms governing $K(z,+)$ should be examined as depth z *decreases*, this being the natural

direction of flow of the upward stream. Finally, owing to the negative sign in front of $K(z,+)$, the signs of the increments in $K(z,+)$ and $D(z,+)$ are *opposite*. Thus there is what we may term as a *contravariation* in the magnitudes of $K(z,+)$ and $D(z,+)$. This is the final distinction that must be made between the two streams, for the present.

Whence does this striking difference in the relative variations of the magnitudes of $K(z,+)$ and $D(z,+)$ arise? In what light should this difference be viewed? The answer to the first question is that the difference arises in the definitions of $K(z,+)$ and $K(z,-)$; each is defined by writing:

$$\text{"}K(z,\pm)\text{"} \quad \text{for} \quad -\frac{1}{H(z,\pm)}\frac{dH(z,\pm)}{dz} \quad .$$

Now a plane-parallel medium representing a natural hydrosol, by its very physical nature and usual coordinate system, normally invites the choice of the downward direction as the direction of increasing z values. Thus the spatial evolution of quantities associated with the downwelling stream are treated in a natural way, i.e., so that the natural unfolding of radiant energy in a downward direction takes place in the direction of the natural unfolding of the coordinate system, i.e., along with increasing z coordinates. The upwelling stream on the other hand naturally evolves spatially in the direction of decreasing z values, hence the contravariation, or topsy turvy interdependence of $K(z,+)$ and $D(z,+)$. This contravariation therefore is not an essential phenomenon and so can be erased and converted to a covariation if we reinterpret the derivative dH/dz as $dH/d(-z)$ when considering the upward-flowing case. In answer to the second question, all that can reasonably be done is to view this state of affairs as an inessential perversity of standard coordinate systems, and to understand that it is the inevitable result of an attempt to depict an inherently three-dimensional process by an artificial two-dimensional symbolism designed by a basically one-dimensional thought process.

A Covariation Rule of Thumb

The general law (28) governing the interdependence of $K(z,-)$ and $D(z,-)$, while of extreme of importance in establishing the exact relationship between these two quantities, is somewhat unwieldy for use in quick estimates of the relative magnitudes of their increments. We conclude the present section with the derivation of a simple rule of thumb, based on experimental evidence, which relates in a linear manner the relative magnitudes of $K(z,-)$ and $D(z,-)$, and also their increments. We begin with the exact expression for $R(z,-)$ in terms of $K(z,\pm)$ and $a(z,\pm)$ as given in (25) of Sec. 9.2:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)} \quad .$$

Here $a(z,\pm) = a(z)D(z,\pm)$. Now it is an experimental fact that the difference between the magnitudes of $K(z,+)$ and $K(z,-)$ over the visible spectrum is very small for many *practical* purposes, being on the order of five percent of the magnitude of $K(z,-)$. (This may be seen in Sec. 10.4.) Hence we may replace $K(z,+)$ by $K(z,-)$ in the preceding formula and solve for $K(z,-)$. The result is:

$$\boxed{K(z,-) = a(z)\left[\frac{R(z,-)D(z,+) + D(z,-)}{1 - R(z,-)}\right]} \quad . \tag{36}$$

Equation (36) may be taken, as it stands, as a rule of thumb connecting $K(z,-)$ with $a(z)$. This equation serves to underscore the conclusion reached earlier in this work that $K(z.-)$ is basically an absorption-like optical property of a medium.

To arrive at the desired rule of thumb we now make use of the experimental-numerical relations (33) and (34) between $D(z,+)$ and $D(z,-)$ and also of the fact that $R(z,-)$ is of the order 0.02. The result is:

$$\begin{aligned} K(z,-) &= a(z)\left[\frac{R(z,-)\dfrac{D(z,+)}{D(z,-)} + 1}{1 - R(z,-)}\right] D(z,-) \\ &= a(z)\left[\frac{\dfrac{1}{50} \times 2 + 1}{1 - \dfrac{1}{50}}\right] D(z,-) \\ &= \frac{52}{49}\, a(z)D(z,-) = 1.06\, a(z)D(z,-) \quad . \end{aligned}$$

Hence:

$$\boxed{\Delta K(z,-) = 1.06\, a(z)\Delta D(z,-)} \quad , \tag{37}$$

which is the desired rule of thumb relating the covariation of $K(z,-)$ and $D(z,-)$.

9.4 General Analytic Representation of the Observable Reflectance Function

The concept studied in this section is the observable reflectance function $R(\cdot,-)$ whose value at a depth z in an arbitrarily stratified plane-parallel optical medium is given by:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad ,$$

where, as usual, the quantities $H(z,\pm)$ are the observed upwelling (+) and downwelling (-) irradiances at depth z in the medium (re: (16) of Sec. 9.2). Several representations of the function $R(\cdot,-)$ are established which will, (a)

explicitly exhibit in terms of differential equations and definite integrals the dependence of $R(\cdot,-)$ on the *inherent* optical properties of the medium, as far as this is possible; (b) illustrate the dynamic equilibrium-seeking tendency of $R(\cdot,-)$ which appears to hold in all plane-parallel media; and finally, (c) suggest some methods of solving the problem of predicting the depth-structure of $R(\cdot,-)$ in general media. To place the present discussions in their proper perspective for the general reader, we prefix the following observations.

The reflectance function $R(\cdot,-)$ is one of a set of seven main *apparent* optical properties introduced in Sec. 9.2. This set consists of the functions $R(\cdot,\pm)$, $K(\cdot,\pm)$, $k(\cdot,\pm)$, and k, and is defined along with $D(\cdot,\pm)$ in terms of the four directly observable radiometric functions: $H(\cdot,\pm)$, $h(\cdot,\pm)$, where $h(\cdot,\pm)$ are the upwelling (+) and downwelling (-) scalar irradiance functions. The theory of the measurement of these latter radiometric quantities and a discussion of the salient physical characteristics of this extremely useful set of apparent optical properties was briefly sketched in Sec. 2.7 (see Fig. 2.18). Further discussion of these properties is given in Chapter 13. Section 9.6 contains a classification of the optical properties of an optical medium into the classes of *inherent* and *apparent* optical properties, and the necessary distinctions that must be made between them, in both experimental and theoretical procedures.

The main fundamental set of local inherent optical properties of any scattering-absorbing optical medium consists of the functions α and σ, the volume attentuation and volume scattering function, respectively. These functions are by definition independent of the ambient light field. The apparent optical properties, however, depend jointly on the inherent optical properties and the ambient light field. Specifically, the apparent optical properties depend on α, σ, and the radiance distributions $N(z,\cdot)$ in the medium.

Despite this dependence of the R, K, D, and k functions on ephemeral lighting conditions, they exhibit a behavior in both space and time of such a strikingly regular and generally predictable kind, that each is dignified with the appellation: "optical property." However, we point out the fact that this is a matter of first appearances only, and that, under incisive analytical and experimental scrutiny, their regularities are seen to be at the mercy of variable boundary lighting conditions and the internal distribution of the values of α and σ in an optical medium. To emphasize this fact, the qualification "apparent" has been put before "optical property."

Detailed examples of the regular behavior of the apparent optical properties are given in Secs. 9.2 and 9.3, in the following sections, and in the remaining chapters of Part III. The present section adds to this store of knowledge of the apparent optical properties by developing in detail the exact differential and integral representations of the reflectance function $R(\cdot,-)$ and drawing some theoretical and practical conclusions from them.

The Differential Equation for R(·,-): Unfactored Form

The physical setting for the derivations that follow is a plane-parallel source-free optical medium whose inherent optical properties spatially depend only on depth, and this depth dependence is assumed arbitrary. Further, the medium may have either finite or infinite optical depth with arbitrary boundary reflectance properties and arbitrary incident lighting conditions on its upper and lower boundaries.

Assume that an irradiance probe sweeps through a range of depths between z' and z'', where $0 \le z' \le z'' \le \infty$, and that at each of the depths in this range readings $H(z,\pm)$ are taken, $z' \le z \le z''$. Then the reflectance $R(z,-)$ at each depth is determined by:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad ,$$

and its depth rate of change $dR(z,-)/dz$ for downward motion or for upward motion is easily found and is given as (re: (32) of Sec. 9.2):

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad .$$

We may now introduce the exact representations of $K(z,\pm)$ established earlier ((18) and (19) of Sec. 9.2):

$$\mp K(z,\pm) = [a(z,\pm) + b(z,\pm)] - b(z,\mp)R(z,\pm) \quad ,$$

where $a(z,\pm) = a(z)D(z,\pm)$, and $b(z,\pm)$ are the values of the absorption and backward scattering functions at depth z for the upwelling (+) and downwelling (-) streams of radiant flux. Substituting these representations of $K(z,\pm)$ in the above derivative, the result is:

$$\boxed{\begin{gathered} -\frac{dR(z,-)}{dz} = b(z,+)R^2(z,-) - c(z)R(z,-) + b(z,-) \quad , \\ \text{where} \\ c(z) = [a(z,-) + a(z,+) + b(z,-) + b(z,+)] \quad . \end{gathered}} \qquad (1)$$

This is the desired differential equation for R(·,-) in *unfactored form*, and is the basic differential equation governing the *observable* reflectance function. It is an exact equation within the presently chosen general physical setting and forms the basis for all our subsequent deductions of the properties of R(·,-). The mathematical structure of (1) is that of a general Riccati equation, which generally has nonelementary solutions. The reader should not fail to observe the striking resemblance between (1) above and (39) of Sec. 8.7, keeping in mind (11) and (12) of Sec. 8.3.

We shall return to consider this resemblance later, in the *equivalence theorem*.

The Differential Equation for $R(\cdot,-)$: Factored Form

The basic differential (1) may be factored by observing that its right-hand side is a quadratic in $R(z,-)$ for each depth z. Thus for a given z, the roots of the quadratic equation:

$$b(z,+)t^2 - c(z)t + b(z,-) = 0$$

are

$$R_\alpha(z,-) = \frac{c(z) + [c^2(z) - 4b(z,-)b(z,+)]^{1/2}}{2b(z,+)} ,$$

$$R_q(z,-) = \frac{c(z) - [c^2(z) - 4b(z,-)b(z,+)]^{1/2}}{2b(z,+)}$$

Hence (1) may be written:

$$\boxed{-\frac{dR(z,-)}{dz} = b(z,+)[R(z,-) - R_\alpha(z,-)][R(z,-) - R_q(z,-)]} . \tag{2}$$

Equation (2) is the *factored form* of the differential equation for $R(\cdot,-)$. The function $R_q(\cdot,-)$ is the *equilibrium function* for $R(\cdot,-)$, and $R_\alpha(\cdot,-)$ is the *attenuation function* for $R(\cdot,-)$. The kinship of $R_\alpha(\cdot,-)$ and $R_q(\cdot,-)$ with the two-D theory's k_+ and k_-, as given in (12) of Sec. 8.5, may be noted. Equation (2) will be cast into deep perspective within radiative transfer theory when we study the universal transport equation in Chapter 11 (see (15) of Sec. 11.2 and (1) of Sec. 11.3).

Second-Order Form of Differential Equation for $R(\cdot,-)$

We now introduce a new function Q defined on the depth interval of interest, and having the property:

$$\frac{Q'(z)}{Q(z)} = b(z,+)R(z,-) ,$$

where the prime denotes differentiation with respect to z.

It follows that Q satisfies a linear homogeneous second-order differential equation whose coefficients are functions of $c(z)$ and $b(z,\pm)$. To see this, we differentiate each side of the above relation. The result is:

$$\frac{Q(z)Q''(z) - [Q'(z)]^2}{Q^2(z)} = b(z,+)R'(z,-) + b'(z,+)R(z,-) .$$

Then using (1) for the expression equivalent to $R'(z,-)$:

$$\frac{Q''(z)}{Q(z)} - \left[\frac{Q'(z)}{Q(z)}\right]^2 = b(z,+)[-b(z,+)R^2(z,-) + c(z)R(z,-) - b(z,-)] + b'(z,+)R(z,-) \quad ;$$

so that:

$$\frac{Q''(z)}{Q(z)} = b^2(z,+)R^2(z,-)$$
$$+ [-b^2(z,+)R^2(z,-) + c(z)R(z,-)b(z,+) - b(z,-)b(z,+)]$$
$$+ b'(z,+)R(z,-) \quad .$$

Hence,

$$\frac{Q''(z)}{Q(z)} = [c(z)b(z,+) + b'(z,+)]R(z,-) - b(z,-)b(z,+) \quad .$$

Using the defining relation for Q once again, this becomes

$$\boxed{Q''(z) - \left[c(z) + \frac{b'(z,+)}{b(z,+)}\right] Q'(z) + b(z,-)b(z,+)Q(z) = 0} \quad . \tag{3}$$

Equation (3) is the desired *homogeneous second order differential equation* for Q. Upon obtaining its solution, the defining relation for Q is used to obtain the expression for $R(\cdot,-)$.

The Equilibrium-Seeking Theorem for $R(\cdot,-)$: Preliminary Observations

We turn now to study one of the deeper properties of the observable reflectance function, which states that *the observable reflectance $R(z,-)$ at each depth z is tending, with increasing depth, to some well-defined equilibrium value, and that it will attain that value if the medium is sufficiently uniform in structure.* To prepare the way for this theorem, known as the *equilibrium-seeking theorem* for $R(\cdot,-)$, we shall make some preliminary observations on the nature of "equilibrium concepts" in general radiative transfer theory.

Equilibrium theorems abound in the theory of radiative transfer. Perhaps the earliest explicit and unmistakable instance of an equilibrium theorem was given by means of Koschmieder's equation (Sec. 4.3) which describes the apparent radiance N_r of an object of inherent radiance N_0 as seen along a horizontal path of length r along which both the inherent optical properties of the medium and the ambient lighting conditions are constant:

$$N_r = N_o\, e^{-\alpha r} + \frac{N_*}{\alpha}\,[1 - e^{-\alpha r}] \quad .$$

Here the quantity N_* (the path function) is the (constant) radiance per unit length of path, in the direction of the path at every point of the path, and which is generated by scattering of the ambient light. The quantity α is the fixed value of the volume attenuation function for the medium along the path of sight. Assuming that this type of path may be extended indefinitely in the direction away from the object, the equation implies that, for any medium with $\alpha > 0$,

$$\lim_{r \to \infty} N_r = \frac{N_*}{\alpha} \quad .$$

The quantity N_*/α, usually denoted by "N_q", is the *equilibrium radiance* of the path of sight, and in this case is simply the observable horizontal radiance. It is dependent on both the local inherent optical properties of the medium (α, σ) and the lighting conditions along the path (N_*). The term *equilibrium radiance* is understood in the following sense: For any initial choice of N_o, N_r tends toward and eventually attains the value N_q. Thus if N_o exceeds N_q, then N_r decreases from N_o to N_q as r goes from 0 to ∞. On the other hand, if N_o is less than N_q, then N_r increases from N_o to N_q as r goes from 0 to ∞.

This phenomenon of the equilibrium-seeking tendency of the apparent radiance actually holds for an arbitrary path of sight in an arbitrary optical medium *along which there are no sources* and along which $\alpha > 0$. This may be seen by taking the general transfer equation for radiance:

$$\frac{dN}{dr} = -\alpha N + N_*$$

and writing it in the form:

$$\frac{dN}{dr} = -\alpha[N - N_q] \quad ,$$

where we have written "N_q" for N_*/α, as in Sec. 4.3. It must be emphasized that this equation is completely general; hence α may change from point to point along a path, N_* (and hence N_q) may depend on direction about a fixed point, and the angular dependences of N_* at two different points may be quite distinct. Now select any path of sight in the medium, and choose an initial point of the path. At this point suppose the value of N is given. If then $N > N_q$, the above equation immediately shows that $dN/dr < 0$, so that N tends toward the value of N_q at this point as r increases. On the other hand, if $N < N_q$, then $dN/dr > 0$, and N tends toward N_q once again as r increases. Now it is quite possible that N_q may change from point to point along the path. But the important fact to observe is that

at every point of the path, regardless of the relative sizes of N and N_q, the tendency of N *at that point* is to change its value so as to *decrease* the absolute value of the difference $N - N_q$ at that point.

The phenomenon of the equilibrium-seeking tendency of radiance in an arbitrary medium gives rise to a host of equilibrium theorems for various other radiometric quantities and even for the apparent optical properties.

These equilibrium theorems are explored in detail in Chapter 11 where it is shown that no less than 34 radiometric and related concepts are subject to a single equilibrium principle.

The following discussion of the equilibrium theorem for $R(\cdot,-)$ is patterned after that exhibited for N above. Furthermore, this special discussion will pave the way for some interesting observations of the properties of the reflectance function as seen in the light of the principles of invariance. These observations will be given later in this section.

The Equilibrium-Seeking Theorem for $R(\cdot,-)$

To establish the present equilibrium theorem for $R(\cdot,-)$, consider an arbitrarily stratified source-free plane-parallel medium over the depth interval $0 \le z' < z'' \le \infty$ in which scattering takes place, i.e., $\sigma \not\equiv 0$. The medium may be optically shallow or deep; its boundary reflectances are arbitrary, as are the boundary lighting conditions. The present setting, therefore, is of maximum generality. Imagine a reflectance meter at depth z in the medium. The reading $R(z,-)$ gives the complete reflectance of the material between the level z and the lower boundary, inclusively. This number is a complex combination of the effects of the standard reflectance of the medium in that depth interval (i.e., the standard reflectance $R(z,z'')$ of the slab $X(z,z'')$), the interreflections between $X(0,z')$ and $X(z',z'')$, and the angular structure of the downwelling incident flux at level z. The angular structure of the downwelling flux at level z in turn depends on the inherent optical properties of the medium throughout its extent. However, despite this complex situation, there exists at every level z along with $R(z,-)$, the values $R_\alpha(z,-)$ and $R_q(z,-)$ of the attenuation function and equilibrium function associated with $R(\cdot,-)$, which guide the evolution of $R(z,-)$ as z increases. In view of the active role played by $R_\alpha(z,-)$ and $R_q(z,-)$ in determining the depth behavior of $R(z,-)$, we pause to examine their structure. It turns out that $R_\alpha(z,-)$ is of central interest. We shall first establish the fact that $R_\alpha(z,-) \ge 1$ for all z.

We deduce the fact that $R_\alpha(z,-) \ge 1$ for all z on strictly analytical grounds, starting from the defining equation:

$$R_\alpha(z,-) = \frac{c(z) + [c^2(z) - 4b(z,-)b(z,+)]^{1/2}}{2b(z,+)} \quad .$$

Observe first that the value $R_\alpha(z,-)$, when considered as determined solely by the magnitude of $c(z)$, monotonically increases with $c(z)$, in the sense that we hold $b(z,\pm)$ fixed and let $c(z)$ increase. Thus in particular if for some special value c_o of $c(z)$ we can show that $R_\alpha(z,-) \geq 1$, then for all $c(z) \geq c_o$ we will certainly have $R_\alpha(z,-) \geq 1$. Now $c(z) = a(z,-) + a(z,+) + b(z,-) + b(z,+)$. Hence (since all a's and b's are nonnegative):

$$c(z) \geq b(z,+) + b(z,-)$$

in fact the strict inequality holds in all real media. Let us denote $b(z,+) + b(z,-)$ by "c_o". Then

$$\frac{c_o + \left[c_o^2 - 4b(z,-)b(z,+)\right]^{1/2}}{2b(z,+)} = 1 \quad .$$

It follows that $R_\alpha(z,-) \geq 1$ for $0 \leq z \leq z''$, in every plane-parallel optical medium.

We recall at this point the fact that:

$$R(z,-) \leq 1$$

in all real optical media. (See (29) of Sec. 9.2; in fact the strict inequality holds in such media.) From these inequalities we deduce the fact that the difference $R(z,-) - R_\alpha(z,-)$ in all real media is negative for all z.

Continuing with the development of the theorem, suppose that we now measure $R(z,-)$, $z > 0$, and then move the reflectance meter a small distance in the *upward* direction (maintaining, of course, its horizontal collection-orientation throughout the move). What we are in effect doing by such a move is increasing by a small amount the material of the medium below the level occupied by the meter. It turns out that this upward motion is the natural direction of motion one should go in order to discern the equilibrium-seeking behavior of $R(\cdot,-)$, just as the natural direction of motion of the observer in the equilibrium theorem for N was such that it increased the amount of scattering-absorbing material between the observer and the initial point of the path (Figs. 9.2-9.3). In this connection see also the discussion of the contravariation of $K(z,+)$ and $D(z,+)$ presented in Sec. 9.3. Therefore, to analytically describe the result of this motion the derivative term of equation (2) is now read as $dR(z,-)/d(-z)$.

The final steps in in the proof may now readily be taken. Suppose that $R(z,-) < R_q(z,-)$ at the depth under consideration. (See Figs. 9.2, 9.3.) Hence, $R(z,-) - R_q(z,-)$ is negative. By the preceding observations, it is known that $R(z,-) - R_\alpha(z,-)$ is invariably negative in all real media. Thus the derivative $dR(z,-)/d(-z)$ is *positive*, indicating that $R(z,-)$ tends toward the value of the equilibrium reflectance $R_q(z,-)$ *at this depth*, as z decreases. On the other hand, if $R(z,-) > R_q(z,-)$, then $R(z,-) - R_q(z,-)$ is positive, and since $R(z,-) - R_\alpha(z,-)$ is invariably negative it follows that in this case $dR(z,-)/d(-z) < 0$, so that

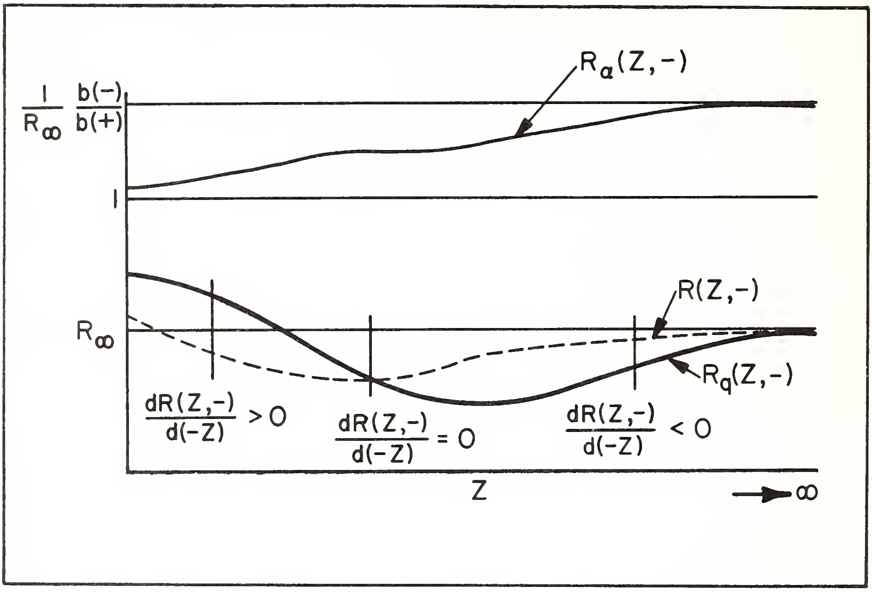


FIG. 9.2 As one moves upward through the hydrosol, in the direction of *decreasing* z , observe how the slope of R(z,-) is always directed so as to decrease the gap between R(z,-) and the equilibrium reflectance $R_q(z,-)$. This is the process referred to by the equilibrium-seeking theorem for R(z,-).

once again R(z,-) tends toward $R_q(z,-)$. This completes the proof of the theorem.

We summarize the equilibrium-seeking theorem symbolically as follows:

$$\text{sign}\left[\frac{dR(z,-)}{d(-z)}\right] = \text{sign}\left[R_q(z,-) - R(z,-)\right] \tag{4}$$

We may now make several observations on this equilibrium-seeking property of R(·,-).

Observation 1

By returning to the basic premises of the present discussion, we observe that the condition $\sigma \not\equiv 0$ was imposed. This condition has both physical and mathematical relevance to the conclusion (4). Mathematically, $R_\alpha(z,-)$ and $R_q(z,-)$ are *prima facie* undefined for the case $\sigma \equiv 0$. Physically, the reflectance of a purely absorbing medium or a vacuum is

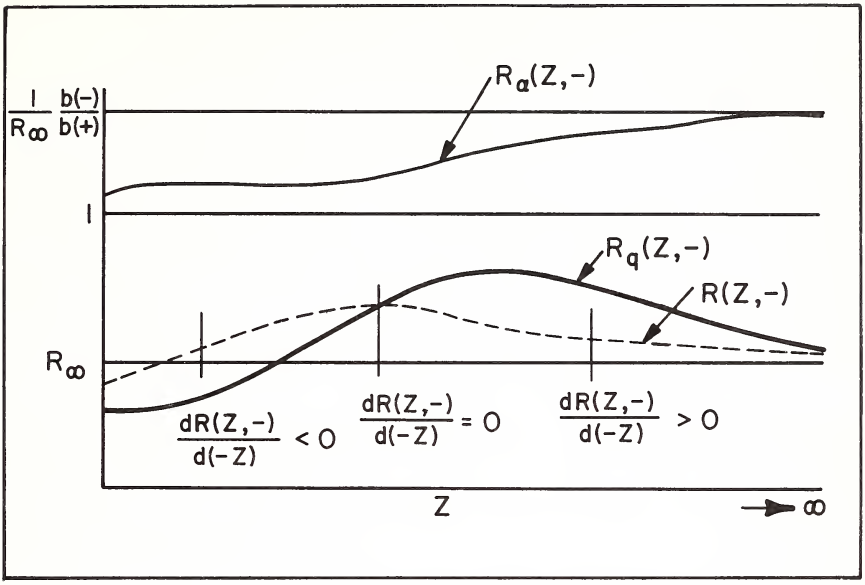


FIG. 9.3 Further variations on the equilibrium-seeking theorem for $R(z,-)$.

trivially zero. To lift the veil of mathematical indeterminacy of $R(z,-)$ in the case of $\sigma \equiv 0$, we return to equation (1). Under the present conditions (1) reduces to:

$$\frac{dR(z,-)}{d(-z)} = -\,c(z)R(z,-) \quad ,$$

where in this case $c(z) = a(z,-) + a(z,+) > 0$. Hence if the initial value of $R(\cdot,-)$ at z'' is $R(z'',-) \geq 0$, then clearly:

$$R(z,-) = R(z'',-)\exp\left\{-\int_z^{z''} c(z')dz'\right\} \quad .$$

By letting $b(z,\pm) \to 0$ in the quadratic equation governing $R_q(z,-)$ we see that $R_q(z,-) = 0$. The preceding formula for $R(z,-)$ shows that, for media with $a(z) > 0$ for all z, and $\sigma \equiv 0$ on $[z',z'']$, $\lim_{[z''-z]\to\infty} R(z,-) = 0$. Hence the equilibrium seeking tendency of $R(\cdot,-)$ is borne out for this case also.

Observation 2

What about the opposite case to that just considered? Namely that $\sigma(z;\xi';\xi) \neq 0$ for all z,ξ',ξ, and $a \equiv 0$? It follows that $R_\alpha(z,-) = 1$ and that $R_q(z,-) = b(z,-)/b(z,+)$

for each z. Now suppose that $R(z,-) < 1$ at some depth z in the interval $[z',z'']$. This implies that $H(z,+) < H(z,-)$ and hence that the radiance distribution $N(z,\cdot)$ over Ξ_+ (the upward directions) is on the average less than the radiance distribution over Ξ_- ,(the downward directions). Now using the scattering functions defined in (31) and (32) of Sec. 9.3 and the definitions of $b(z,\pm)$ as given in (8) of Sec. 8.3, we see that in general,

$$\frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} = \frac{\int_{\Xi_-} \sigma_{-+}(z;\xi')N(z,\xi')d\Omega(\xi')}{\int_{\Xi_+} \sigma_{+-}(z;\xi')N(z,\xi')d\Omega(\xi')} \quad .$$

But if the preceding supposition, namely that $R(z,-) < 1$, holds, then it follows that:

$$\frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} > 1 \quad ,$$

or in other words:

$$R(z,-) < \frac{b(z,-)}{b(z,+)} = R_q(z,-) \quad .$$

By the equilibrium-seeking theorem, it follows that $dR(z,-)/d(-z)$ is *positive*. Hence as depth is *decreased* in the purely scattering medium, the values $R(z,-)$ *increase* monotonically. But since the preceding supposition, namely $R(z,-) < 1$, was for an arbitrary $R(z,-)$ magnitude less than unity, it follows from (2) that for every z,

$$\lim_{(z''-z)\to\infty} R(z,-) = 1 \quad .$$

Thus we have in observations 1 and 2 above proved (or outlined proofs for) some outstanding folklore about the elementary properties of $R(\cdot,-)$ in plane-parallel media. These proofs were arrived at by reasoning strictly from the various exact differential equations governing $R(\cdot,-)$. In this way we hope to illustrate the power inherent in that approach to radiative transfer problems under development in this chapter, which discards particular mathematical models and which concentrates on the study of *directly observable* quantities of the light field. It is to be emphasized that the reasoning in this approach proceeds directly from the *exact forms* of the equations of transfer.

The Integral Representations of R(z,-)

Starting with the factored form (2) of the transport equation for $R(\cdot,-)$ we make use of the separation of variables that exists within it, and we write:

$$\frac{dR(z,-)}{[R(z,-) - R_\alpha(z,-)][R(z,-) - R_q(z,-)]} = b(z,+)d(-z) \quad .$$

If we formally integrate each side, we obtain the desired integral representation by letting $-z$ range from some value z_1, to $z_2 > z_1$. Let the corresponding values of $R(\cdot,-)$ at these depths be distinct: $R(z_1,-) \neq R(z_2,-)$; the depth range of interest can always be partitioned into pieces so that this is true; otherwise the problem of $R(\cdot,-)$ is trivial over this depth range. In other words, the range of integration can be subdivided into intervals so that over each, $R(\cdot,-)$ is monotonic and so that there is a one to one correspondence between the values of $R(\cdot,-)$ and the points of the interval. With these observations we may then write:

$$\boxed{\int_{R(z_1,-)}^{R(z_2,-)} \frac{dt}{[t - R_\alpha(t\ -)][t - R_q(t\ -)]} = \int_{z_1}^{z_2} b(t,+)dt} \quad , (5)$$

which is the desired *integral representation of* $R(\cdot,-)$. The variable t in the integrals acts as a dummy variable of integration, oriented as in the equilibrium-seeking theorem.

An alternate integral representation of $R(\cdot,-)$ may be obtained from (1) in which the variables are also conveniently separated. The same general arguments used to establish (5) may now be directed to the equation (1). The result is:

$$\boxed{\int_{R(z_1,-)}^{R(z_2,-)} \frac{dt}{b(t,+)t^2 - c(t)t^2 + b(t,-)} = (z_2 - z_1)} \quad . (6)$$

Applications

We now discuss two methods of evaluating $R(\cdot,-)$ by means of its differential and integral equation representations given above. We illustrate the use of (5) for a very simple case, which is a useful approximation to reality, namely the case in which $R_\alpha(\cdot,-)$ and $R_q(\cdot,-)$ are constant functions. The second method is based directly on (1) or (3) and promises to yield a means of determining $R(\cdot,-)$ under realistic conditions.

Special Closed Form Solution

If over some depth interval the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ are constant, then the functions $R_\alpha(\cdot,-)$ and $R_q(\cdot,-)$ are constant functions over the same arbitrary depth interval,

say $[z_1, z_2]$, $z_2 > z_1$, in a plane-parallel medium. Under these conditions, (5) is immediately integrable, and the definite integrals take the forms:

$$\frac{1}{R_q - R_\alpha}\left[\ln \frac{t-R_q}{t-R_\alpha}\right]_{R(z_1,-)}^{R(z_2,-)} = b(+)(z_2-z_1) \quad .$$

From the definitions of R_q and R_α, we see that $R_\alpha - R_q > 0$ and in fact:

$$R_\alpha - R_q = \frac{[c^2 - 4b(-)b(+)]^{1/2}}{b(+)} \quad ,$$

where we have written:

$$\text{"}c\text{"} \quad \text{for} \quad a(-) + a(+) + b(-) + b(+) \quad ,$$

and where $a(\pm)$ and $b(\pm)$ are the assumed constant values of the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ over the depth range $[z_1, z_2]$. In the present method all four of these quantities may be distinct.

Applying the limits to the left integral, we have:

$$\ln\left[\frac{R(z_2,-)-R_q}{R(z_2,-)-R_\alpha}\right] = \ln\left[\frac{R(z_1,-)-R_q}{R(z_1,-)-R_\alpha}\right] - [c^2-4b(-)b(+)]^{1/2}(z_2-z_1).$$

Hence,

$$\left[\frac{R(z_2,-)-R_q}{R(z_2,-)-R_\alpha}\right] = \left[\frac{R(z_1,-)-R_q}{R(z_1,-)-R_\alpha}\right] \exp\left\{- [c^2-4b(-)b(+)]^{1/2}(z_2-z_1)\right\} .$$

If we write:

$$\text{"}c(z_1,-)\text{"} \quad \text{for} \quad \frac{R(z_1,-) - R_q}{R(z_1,-) - R_\alpha} \quad ,$$

then:

$$\boxed{R(z_2,-) = \frac{R_q - R_\alpha c(z_1,-)\exp\{-[c^2-4b(-)b(+)]^{1/2}(z_2-z_1)\}}{1 - c(z_1,-)\exp\{-[c^2-4b(-)b(+)]^{1/2}(z_2-z_1)\}}} \quad (7)$$

Hence if the four constants $a(\pm)$ and $b(\pm)$ are known or estimable over an interval $[z_1, z_2]$ and $R(z_1,-)$ is known, then $R(z_2,-)$ is determinable. Observe that if we let $(z_2-z_1) \to \infty$, thereby simulating an infinitely deep layer, then $R(z_2,-) \to R_q$. Hence R_q in this instance is the R_∞ - quantity of the classical theory.

The points of contact with the classical theory may be increased by observing that if we set: $a(+) = a(-) = a^*$, and $b(+) = b(-)\ b^*$, then:

$$[c^2 - 4b(-)b(+)]^{1/2} = 2[a^* + 2b^*)]^{1/2}$$
$$= 2k \qquad ,$$

where k is the diffuse absorption coefficient of the classical one-D model of the two-flow theory (see, e.g., (8) or (32) of Sec. 8.6).

By partitioning an inhomogeneous medium into essentially homogeneous contiguous layers, successive applications of (7) will yield a useful practical formula for the reflectance of the entire medium. The solution (7) automatically includes the effects of interreflections between the partition pieces. Thus suppose the medium, which extends over an interval $[z_o, z_n]$, is partitioned in n homogeneous layers defined by the depths: $[z_o, z_1]$, $[z_1, z_2]$, ..., $[z_{n-1}, z_n]$. If $R(z_n,-)$ is known (this may be the reflectance of the bottom boundary of the layer $[z_{n-1}, z_n]$), then by (7) we find $R(z_{n-1},-)$. Another application of (7) with $R(z_{n-1},-)$ as the initial reflectance then yields $R(z_{n-2},-)$, and so on to $R(z_o,-)$ which then is the reflectance associated with the medium over the depth interval $[z_o, z_n]$.

Differential Analyzer or Digital Solutions

Equations (1) and (3) as they stand, are suitable for determinations of $R(\cdot,-)$ by means of differential analyzer (and analog) or digital techniques, especially when the functions $a(\cdot,\pm)$ and $b(\cdot,\pm)$ vary extensively over the medium.

Series Solutions

By means of series solution techniques, equations (1) and (3) may also be used to solve the difficult problem of determining $R(\cdot,-)$ over some interval $[z_1, z_2]$ when $a(\cdot,\pm)$ and $b(\cdot,\pm)$ are nonconstant and known over this interval. By expanding the coefficients of $R^2(z,-)$ and $R(z,-)$, and the $b(z,-)$ term in (1) in terms of infinite series in z, recursion formulas may be obtained for the coefficients in the infinite series expansion of $R(z,-)$ over $[z_1, z_2]$.

Equivalence Theorem for $R(\cdot,-)$

Comparison of the differential equation (1) with (39) of Sec. 8.7 shows that the observable reflectance function $R(\cdot,-)$ and the standard reflectance function $R(\cdot,z_1)$, both defined in a given slab $X(0,z_1)$, satisfy the same differential equation within $X(0,z_1)$. This is a somewhat arresting fact since the interpretation of the two numbers $R(z,-)$ and $R(z,z_1)$ are quite different conceptually. Briefly,

$R(z,-)$ is obtained as a ratio of $H(z,+)$ to $H(z,-)$ deep within $X(0,z_1)$; and $R(z,z_1)$ is the ratio of $H(z,+)$ to $H(z,-)$ when $X(z,z_1)$ is thought of as an *isolated* slab of $X(0,z_1)$. This is not to assert that these functions need always agree in value at each point. Indeed, for $z = z_1$, we have $R(z_1,z_1) = 0$, since the reflectance of a slab of zero thickness is zero. On the other hand $R(z_1,-)$ is generally not zero: its magnitude being the reflectance of the lower boundary of $X(0,z_1)$. However, if $X(0,z_1)$ has no lower boundary and no sources there or one whose reflectance and transmittances are 0 and 1, then $R(z_1,-) = 0$ and we would expect $R(z,-) = R(z,z_1)$ for every z in $[0,z_1]$, since both functions now satisfy not only the same differential equation but also the same initial condition.

The principles of invariance for irradiance help clarify this somewhat unexpected relation between $R(z,-)$ and $R(z,z_1)$. From (1) of Sec. 8.1 we have for the present medium $X(z,z_1)$:

$$H(z,+) = H(z_1,+)T(z_1,z) + H(z,-)R(z,z_1)$$

If $X(z,z_1)$ has no upward irradiance at level z_1, so that $H(z_1,+) = 0$, then:

$$H(z,+) = H(z,-)R(z,z_1)$$

on the other hand, we have, by definition of $R(z,-)$:

$$H(z,+) = H(z,-)R(z,-)$$

whence follows the equality of the two functions $R(\cdot,-)$ and $R(\cdot,z_1)$ over $[0,z_1]$. We shall summarize these observations as follows:

Equivalence theorem for reflectances: Let $X(0,z_1)$ be an arbitrary stratified source-free plane-parallel optical medium with arbitrary boundary irradiances $H(0,-)$ and $H(z_1,+)$. Then the observable reflectance function $R(\cdot,-)$ and the standard reflectance function $R(\cdot,z_1)$ for a general subslab $X(z,z_1)$ of $X(0,z_1)$, satisfy the same differential equation [(1) above, or (39) of Sec. 8.7]. If $H(z_1,+) = 0$, then $R(z,-) = R(z,z_1)$ for every z in $[0,z_1]$.

The reader may gain still further insight into the connections between $R(\cdot,-)$ and $R(\cdot,z_1)$ by contemplating the connections between $R(\cdot,-)$ and the complete reflectance functions $\mathcal{R}(z,z',z_1)$, $z \le z' \le z_1$ in $X(0,z_1)$; and also the relation (35) of Sec. 7.5.

Connections with the Two-Flow Theory

The equivalence theorem cited above permits a simple bridge to be constructed between the two-flow theories of Chapter 8 and the directly observable quantities $H(z,\pm)$ of the present chapter. The classical one-D model of the two-flow theory of the light field describes the irradiances in a boundaryless, sourceless, isotropically scattering

homogeneous slab over an interval $[0,z_1]$ irradiated at the upper level ($z = 0$) by a directionally uniform radiance distribution and with $H(z_1,+) = 0$ (Sec. 8.6). The theory proceeds on the assumption that $b(z,-) = b(z,+) \equiv b^*$ and $a(z,-) = a(z,+) \equiv a^*$ (i.e., that the backward scattering and absorption functions for each stream are pairwise identical and have the constant starred values over the slab). We can immediately deduce the values $R(0,z)$ and $T(0,z)$ associated with this slab on the basis of the present general theory. To do this, we recall the statement of the equivalence theorem for reflectance equations proved above. This allows us to use the expression for $R(z_2,-)$ given in (7). We need only observe that, under the present setting, we should make the following pairings of variables:

$$z_1 \longleftrightarrow 0$$

$$z_2 \longleftrightarrow z$$

$$R(z_1,-) = 0$$

$$R_\alpha R_q = 1$$

$$R_\alpha + R_q = c/b^*$$

and finally we observe that:

$$[c^2-4b(-)b(+)]^{1/2} = 2[a^*(a^*+2b^*)]^{1/2} = 2k$$

where k is the diffuse absorption coefficient of the one-D model. Hence (7) reduces to:

$$R(0,z_1) = \frac{[1-\exp\{-2kz_1\}]}{R_\alpha - R_q\exp\{-2kz_1\}} \tag{8}$$

$$= \frac{b^* \sinh kz_1}{(a^*+b^*)\sinh kz_1 + k \cosh kz_1} \tag{9}$$

Equation (9) gives the usual form for the reflectance of a slab of depth z_1. (As a check, let $z_1 \to \infty$, and compare result with (8) of Sec. 9.2.)

The remaining form for $T(0,z_1)$ can now be deduced immediately from (42) of Sec. 8.7, but the point of this discussion has essentially been made: The classical two-flow theory is an elementary special degenerate case of the present theory of *directly observable* quantities in *real* light fields. For convenience of reference, the transmittance $T(0,z_1)$ is given by:

$$T(0,z_1) = \frac{k}{(a^*+b^*)\sinh kz_1 + k \cosh kz_1} \tag{10}$$

Summary

This section develops several differential and integral formulas governing the observable reflectance function $R(\cdot,-)$. Methods of using the formulas are outlined. Thus $R(\cdot,-)$ may be obtained directly without first solving for the irradiance functions $H(\cdot,\pm)$ as has been necessary previously. The methods discussed are general enough to allow the determination of $R(\cdot,-)$ if the absorption and backward scattering functions $a(\cdot,\pm)$, $b(\cdot,\pm)$, respectively, for each stream in an arbitrarily inhomogeneous stratified medium are known. A general equilibrium-seeking theorem for $R(\cdot,-)$ is also demonstrated, the substance of which is the fact that the derivative of $R(\cdot,-)$ at each depth z invariably has an algebraic sign so as to decrease the absolute magnitude of the difference $R(z,-) - R_q(z,-)$ between $R(z,-)$ and the value $R_q(z,-)$ of the equilibrium reflectance function. An equivalence theorem is proved which shows that the observable reflectance function $R(\cdot,-)$ and the generally nonobservable standard reflectance function $R(\cdot,z_1)$ for slabs within $X(0,z_1)$ both obey the same differential equation, thereby establishing an important link between these theoretical $(R(\cdot,z))$ and empirical $(R(\cdot,-))$ concepts.

9.5 The Contrast Transmittance Function

The list of transmittance concepts associated with a path $\mathcal{P}_r(z,\xi)$ in an optical medium will be completed in this section with the introduction of a companion transmittance concept to the beam transmittance function introduced in (3) of Sec. 3.10, and the radiance transmittance function introduced in (13) of Sec. 4.5. This new transmittance concept is called the *contrast transmittance function.*

By way of introduction to the contrast transmittance function, we review briefly the general types of transmittance functions studied so far in the present work. Suppose X is an arbitrary optical medium with boundary S. Then associated with X itself is the standard $\mathcal{J}$-operator: $\mathcal{J}(X;a,b)$, introduced in Sec. 3.8, whose physical significance is depicted in (a) of Fig. 9.4. When the incidence region a on the boundary S coincides with the response region b, then $\mathcal{J}(X;a,a)$ is interpreted as a general reflectance operator. When a and b are disjoint, $\mathcal{J}(X;a,b)$ is interpreted as a general transmittance operator. Three general transmittance operators may be associated with the light field in X:

$$\mathcal{J}^{\circ}(X;a,b) \qquad \mathcal{J}^{*}(X;a,b) \qquad \mathcal{J}(X;a,b)$$

The circled and starred operators are associated with transmitted residual and diffuse radiance respectively; the unadorned $\mathcal{J}$-operator describes the undecomposed or directly observable light field. In Sec. 3.8 $\mathcal{J}(X;a,b)$ was defined in detail. $\mathcal{J}^{\circ}(X;a,b)$ is manufactured readily using the geometric structure of X, a Dirac-delta function, and the beam transmittance function T_r for X, much in the way $T^{\circ}(x,z)$

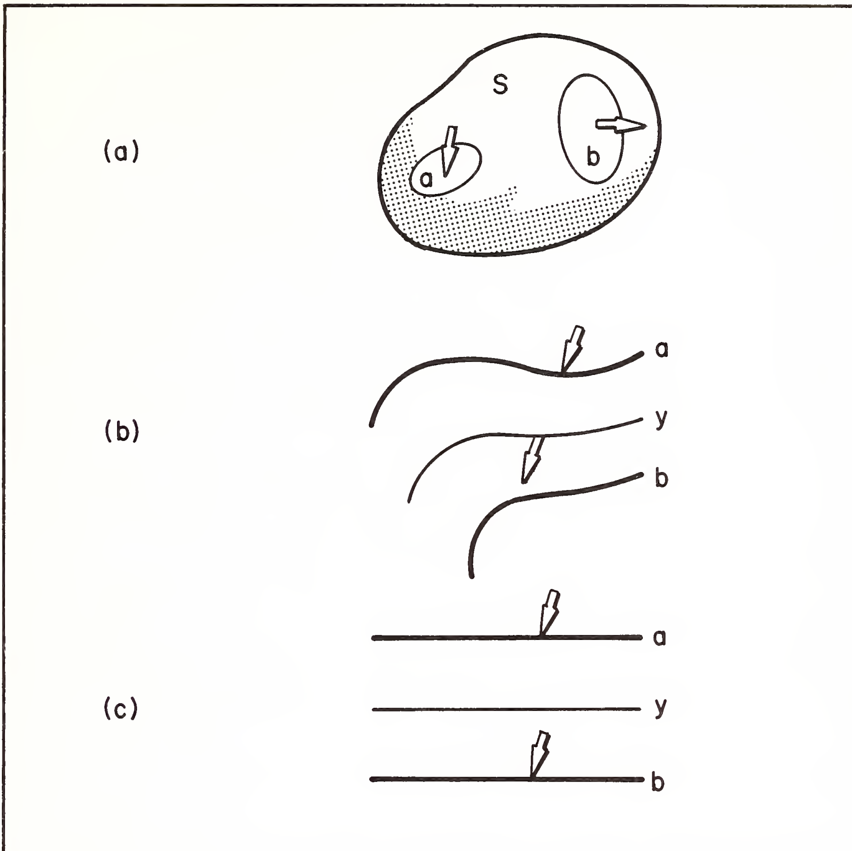


FIG. 9.4 Three settings for transmission operators: (a) general media, (b) one-parameter media, (c) plane-parallel media.

is defined in (32) of Sec. 7.1 for plane-parallel media. $\mathscr{J}^*$ follows by subtraction of $\mathscr{J}^\circ$ from $\mathscr{J}$ (cf. (41) of Sec. 7.1). It follows that:

$$\mathscr{J}(X;a,b) = \mathscr{J}^\circ(X;a,b) + \mathscr{J}^*(X;a,b) \quad . \qquad (1)$$

Much in the same way the complete transmittance operator $\mathscr{T}(x,y,z)$ associated with a subset $X(x,z)$ of a one parameter medium $X(a,b)$ can be rendered into its residual and diffuse parts:

$$\mathscr{T}(a,y,b) = \mathscr{T}^\circ(a,y,b) + \mathscr{T}^*(a,y,b) \quad , \qquad (2)$$

and which is depicted in (b) of Fig. 9.4. Similarly, the decomposition of the standard transmittance operator:

$$T(a,b) = T^{o}(a,b) + T^{*}(a,b) \tag{3}$$

is but a special case of (1) or (2) and is depicted in (c) of Fig. 9.4.

All of the preceding examples pertain to transmittances of three dimensional subsets of an optical medium. One-dimensional subsets can be endowed with similar sets of transmittances. Thus if $\mathcal{P}_r(x,\xi)$ is a path in X, we have in our earlier work associated with it the beam transmittance $T_r(x,\xi)$ (Sec. 3.10). In order to emphasize, during the present discussion, that the beam transmittance is associated with *residual* radiant flux, i.e., flux transmitted over $\mathcal{P}_r(x,\xi)$ without being scattered or absorbed, let us place a circle superscript on the transmittance symbol so: "$T_r^{o}(x,\xi)$." Furthermore, corresponding to $\mathcal{J}(X;a,b)$, $\mathcal{T}(a,y,b)$, and $T(a,b)$ above, that is, corresponding to the transmittance operators for undecomposed (i.e., *observable*) radiance, we have the *radiance transmittance* operator associated with $\mathcal{P}_r(x,\xi)$, as introduced in Sec. 4.5. Let us denote this operator by "$T_r(x,\xi)$." The fundamental roles played by $T_r^{o}(x,\xi)$ and $T_r(x,\xi)$ may be seen by comparing their effects on the initial radiance $N_o(x,\xi)$ of the path $\mathcal{P}_r(x,\xi)$:

$$N_r^{o}(z,\xi) = N_o(x,\xi)T_r^{o}(x,\xi) \tag{4}$$

$$N_r(z,\xi) = N_o(x,\xi)T_r(x,\xi) \tag{5}$$

where $N_r(z,\xi)$ is the radiance at end point z of $\mathcal{P}_r(x,\xi)$ in the direction ξ. We shall consider only straight paths $\mathcal{P}_r(x,\xi)$ in this discussion, so that we have the simple representation: $z = x + r\xi$. We interpret (4) as usual by saying that $N_r^{o}(z,\xi)$ is the residual radiance of $N_o(x,\xi)$ transmitted over $\mathcal{P}_r(x,\xi)$. We interpret (5) in the sense that $T_r(x,\xi)$ is simply the number which when multiplied into $N_o(x,\xi)$ gives $N_r(z,\xi)$ where $N_r(z,\xi)$ and $N_o(x,\xi)$ are the two radiances as they exist at points x and z in the actual light field in X. In other words, the current definition of $T_r(x,\xi)$ consists precisely in agreeing to write:

$$\text{"}T_r(x,\xi)\text{"} \quad \text{for} \quad \frac{N_r(z,\xi)}{N_o(x,\xi)} \tag{6}$$

with respect to a path $\mathcal{P}_r(x,\xi)$ in X. Thus (5) is an elementary consequence of (6). It is obvious that $T_r(x,\xi)$ is immediately obtainable as a consequence of the interaction principle applied to $\mathcal{P}_r(x,\xi)$.

We can now readily round out the roster of transmittance concepts, associated with a path $\mathcal{P}_r(x,\xi)$ in X, and in such a way as to be uniform with the other basic transmittance equations (1)-(3). Thus let us write:

$$\text{"}T_r^{*}(x,\xi)\text{"} \quad \text{for} \quad T_r(x,\xi) - T_r^{o}(x,\xi) \tag{7}$$

Then:

$$\boxed{T_r(x,\xi) = T_r^{o}(x,\xi) + T_r^{*}(x,\xi)} \tag{8}$$

By the work of Sec. 3.11 and Sec. 4.5 we have convenient integral representations of T_r^{o} and T_r. Thus, by (3) of Sec. 3.11:

$$T_r^{o}(x,\xi) = \exp\left\{ - \int_{\mathcal{P}_r(x,\xi)} \alpha \, dr' \right\} \tag{9}$$

and by (2) of Sec. 4.5:

$$T_r(x,\xi) = \exp\left\{ - \int_{\mathcal{P}_r(x,\xi)} \xi \cdot K \, dr' \right\} \quad . \tag{10}$$

In terms of the compact notation of Sec. 4.5, these may also be written:

$$T_r^{o} = T_r [- \alpha] \tag{11}$$

$$T_r = T_r [- \xi \cdot K] \quad . \tag{12}$$

It is clear from the definitions that T_r^{o} is an inherent optical property of $\mathcal{P}_r(x,\xi)$ and T_r (and hence T_r^{*}) is an apparent optical property of $\mathcal{P}_r(x,\xi)$.

The Concept of Contrast

We are now ready to formulate the concept of contrast and develop some of its basic properties. To begin, we can state that the sense in which we use the idea of *contrast* in radiative transfer theory is in the *relative difference* of two radiances. For example if one directs attention to two points 1 , 2 on a distant mountainside or submerged scene which have apparent radiances N_1 and N_2 , then the *contrast of the first point relative to the second* is given quantitatively by the difference quotient: $(N_1-N_2)/N_2$. If $N_1 = N_2$, then there is *zero contrast*, radiometrically speaking between the two points. If it happens that $N_1 > N_2$, then we say that there is a *positive contrast* of point 1 with respect to point 2 . Conversely, since $(N_2-N_1)/N_1$ is now negative, we say that point 2 has *negative contrast* with respect to point 1. One of the fundamental problems in visibility theory in natural optical media is the prediction of the contrast of a visual target against its background. The problem is solved if the contrast transmittance of the path of sight from the observer to the target is known. We shall now develop the theory of the contrast transmittance of an arbitrary path of sight in a natural optical medium.

By extracting the salient features of the intuitive notion of contrast given above, we see that, at its core, radiometric contrast is characterized in terms of two radiances which in turn are associated with two paths of sight in an optical medium. Therefore we can quite generally state the following:

Definition 1. If $\mathcal{P}_r(x_1,\xi_1)$ and $\mathcal{P}_s(x_2,\xi_2)$ are two nontrivial paths (i.e., $r \neq 0$, $s \neq 0$) in an optical medium X (Fig. 9.5), then the difference quotient $[N_r(y_1,\xi_1) - N_s(y_2,\xi_2)]/N_s(y_2,\xi_2)$ is the *apparent contrast* of $N_r(y_1,\xi_1)$ with respect to $N_s(y_2,\xi_2)$. If $r = s = 0$, then the difference quotient is the *inherent contrast* of $N_o(x_1,\xi_1)$ with respect to $N_o(x_2,\xi_2)$.

Regular Neighborhoods of Paths

The notion of contrast attains virtually all its use in situations where the two paths of sight are in some suitable sense optically close neighbors of each other. The following definition isolates the essence of the requisite concept of "closeness" of paths:

Definition 2. Let A and B be two subsets of an optical medium S, and let $C(A,B)$ be a collection of paths in S such that the initial points of the paths are in A and the terminal points of the paths are in B. (See Fig. 9.6.) Then $C(A,B)$ is called a *regular neighborhood* of paths

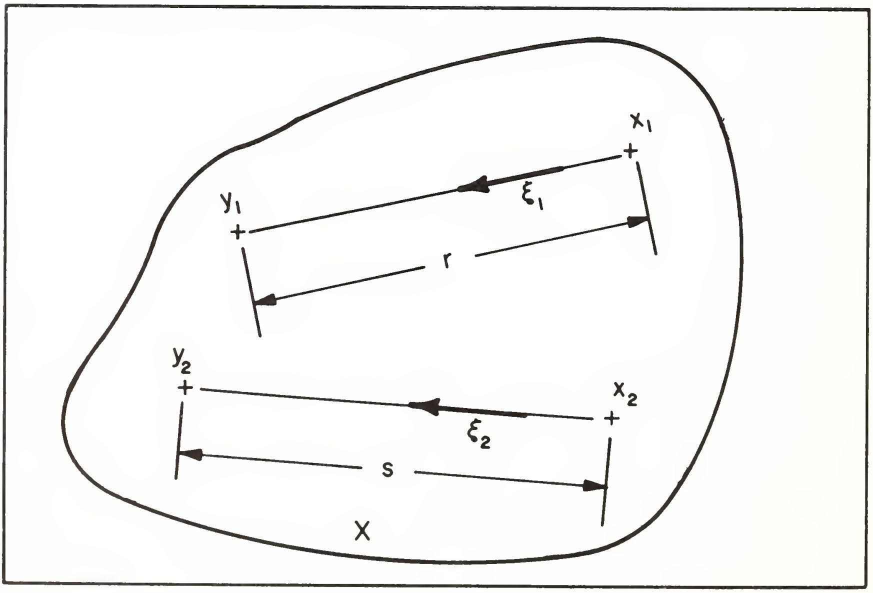


FIG. 9.5 Two paths and their associated contrasts as in definition 1.

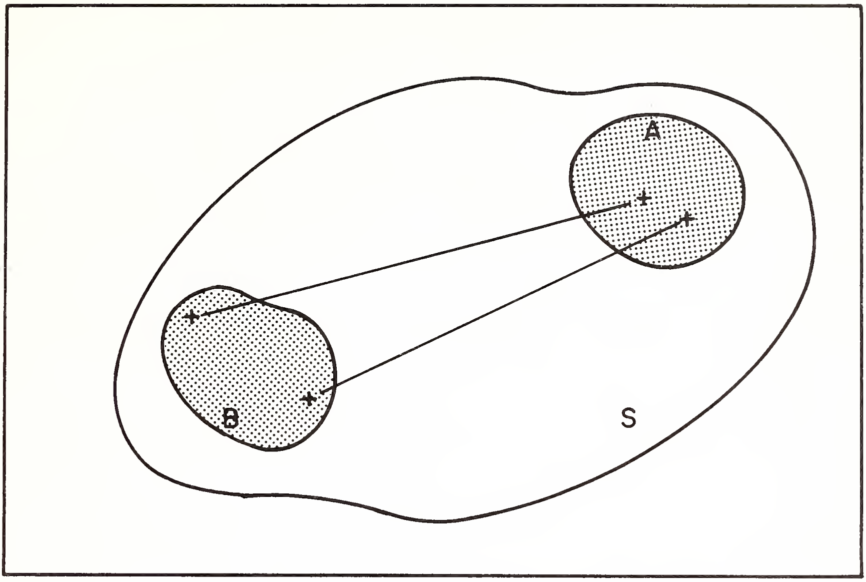


FIG. 9.6 A regular neighborhood of paths.

in S if and only if the paths in C(A,B) have a common length r, a common beam transmittance T_r^o, and a common path radiance N_r^*.

Here are some examples of regular neightborhoods of paths: Let A and B of the definition be the two boundary planes of a stratified plane-parallel medium X(a,b), and let C(A,B) be the set of all paths parallel to a given direction ξ. Then with stratified lighting conditions in X(a,b) C(A,B) is a regular neightborhood of paths in X(a,b). Observe that two paths of C(A,B) may be extremely remote, spatially. As another example, imagine two points x and y in a natural hydrosol, say the sea. It is clear, on intuitive grounds at least, that we can always find two spherical regions A and B or two small parallel plane regions, A and B, about x and y as centers, respectively, such that the associated set C(A,B) of *all* paths with initial points in A and terminal points in B is, for all practical purposes, a regular neighborhood. Our intuition in this matter is grounded in the general appearance of spatial continuity (but not necessarily uniformity) in the inherent optical properties and light fields of real media. Finally, it is clear that if A and B consist of any two single distinct points, then C(A,B) is trivially a regular neighborhood of paths.

The importance of the idea of regular neighborhoods of paths in an optical medium rests in the following observation.

Observation 1. If C(A,B) is a regular neighborhood of paths in an optical medium X, and $\mathcal{P}_{r_1}(x_1, \xi)$ and $\mathcal{P}_{r_1}(x_1, \xi)$ are two paths of C(A,B), with terminal points z_1, z_2, respectively, then:

$$\left[\frac{N_r(z_2,\xi_2) - N_r(z_1,\xi_1)}{N_r(z_1,\xi_1)}\right] = \left[\frac{N_o(x_2,\xi_2) - N_o(x_1,\xi_1)}{N_o(x_1,\xi_1)}\right]\frac{T_r^o(x_1,\xi)}{T_r(x_1,\xi)} \tag{13}$$

or more briefly:

$$C_r(z_1,\xi) = C_o(x_1,\xi)\,\frac{T_r^o(x_1,\xi)}{T_r(x_1,\xi)}$$

where "$C_r(z_1,\xi)$" and "$C_o(x_1,\xi)$" denote the apparent and inherent contrasts occurring in (13). This observation follows readily from use of the fact that if both $N_r(z_2,\xi_2)$ and $N_r(z_1,\xi_1)$ are decomposed:

$$N_r(z_2,\xi_2) = N_r^o(z_2,\xi_2) + N_r^*(z_2,\xi_2)$$

$$N_r(z_1,\xi_1) = N_r^o(z_1,\xi_1) + N_r^*(z_1,\xi_1)$$

and their difference taken, then, since $C(A,B)$ is a regular neighborhood:

$$N_r(z_2,\xi_2) - N_r(z_1,\xi_1) = N_r^o(z_2,\xi_2) - N_r^o(z_1,\xi_1)$$

$$= (N_o(x_2,\xi_2) - N_o(x_1,\xi_1))T_r^o(x_1,\xi)$$

Furthermore:

$$N_r(z_1,\xi_1) = N_o(x_1,\xi_1)T_r(x_1,\xi) \quad ,$$

so that (13) follows. On the basis of this observation, we are led to make:

Observation 2. If $C(A,B)$ is a regular neighborhood of paths in an optical mediuam X, then to each path $\mathcal{P}_r(x,\xi)$ there is assignable the quotient $T_r^o(x,\xi)/T_r(x,\xi)$ where x is in A and r and ξ are such that $x + r\xi$ is in B, and this quotient is an apparent optical property and is generally dependent on x, r and ξ.

Contrast Transmittance and Its Properties

The preceding two observations point up the fact that the quotient of transmittances in (13) is a number which can be assigned to each member of a regular neighborhood $C(A,B)$ of paths. Since the quotient incorporates the apparent optical property $T_r(x,\xi)$, it is also an apparent optical property in general. In view of this and observation 1, we can state:

Definition 3. Let C(A,B) be a regular neighborhood of paths in an optical medium X. The quotient $T_r^o(x,\xi)/T_r(x,\xi)$, where x is in A and x + r is in B, is called the *contrast transmittance* of the path $\mathcal{P}_r(x,\xi)$ in C(A,B), and shall be denoted by "$\mathcal{T}_r(x,\xi)$" or "$\mathcal{T}_r$."

Thus with the preceding definition (13) may be written succinctly as:

$$C_r(z_1,\xi) = C_o(x_1,\xi)\mathcal{T}_r(x_1,\xi) \qquad \text{or} \qquad C_r = C_o\mathcal{T}_r$$

Observation 3. If $\mathcal{P}_r(x,\xi)$ is any path in a general optical medium X, with integrable α and K then:

$$\boxed{\mathcal{T}_r = T_r^o/T_r = T_r[\,-(\alpha - \xi\cdot K)]} \tag{14}$$

The proof is immediate, using (11) and (12) and the multiplicative property (8) of Sec. 4.5. Therefore $\mathcal{T}_r$ is determined for a path of sight once α and K are known over that path. Another result which devolves on the multiplicative property (8) of Sec. 4.5 is:

Observation 4. If $\mathcal{P}_r(x,\xi)$ and $\mathcal{P}_s(y,\xi)$ are any two contiguous subpaths of a path $\mathcal{P}_{r+s}(x,\xi)$ (see Fig. 9.7), and $\mathcal{T}_r$, $\mathcal{T}_s$, and $\mathcal{T}_{r+s}$ are their respective contrast transmittances, then:

$$\boxed{\mathcal{T}_{r+s}(x,\xi) = \mathcal{T}_r(x,\xi)\mathcal{T}_s(x+r\xi,\xi)} \tag{15}$$

or, briefly:

$$\boxed{\mathcal{T}_{r+s} = \mathcal{T}_r\,\mathcal{T}_s}$$

In other words, the contrast transmittance, enjoys the semigroup property along with T_r^o, T_r and the complete transmittance operators $\mathcal{T}(x,y,z)$. The semigroup property (15) holds even for paths along which the index of refraction is nonconstant, and even discontinuous (see, e.g., (25) of Sec. 12.2). The value of (15) lies in the fact that the contrast transmittance of an extended path of sight is determinable once the contrast transmittances of its parts are known.

Some important special cases of (14) are found in the case of stratified plane-parallel media. According to (18) of Sec. 4.5, we have:

$$\mathcal{T}_r = T_r[\,-(\alpha + K\cos\theta)] \tag{16}$$

in such settings. Moreover, if we adopt a two-D model of the light field in an infinitely deep medium X(0,∞), then (16) reduces to:

$$\mathcal{T}_r = \exp\left\{-(\alpha - k_- \cos\theta) r\right\} \tag{17}$$

where K now becomes the diffuse absorption coefficient k_- of the two-D theory (Sec. 8.5). In the case of one-D theory $k_- = -k$ (Sec. 8.6).

<u>Observation 5.</u> If X is an arbitrary optical medium with integrable α and K, then the canonical form of the apparent radiance N_r associated with a path $\mathcal{P}_r$ in X can be written in the form:

$$\boxed{N_r = N_o T_r^o + \frac{N_*}{\alpha - \xi \cdot K} [1 - \mathcal{T}_r]} \tag{18}$$

where $\mathcal{T}_r$ is the contrast transmittance of $\mathcal{P}_r$.

This observation follows at once from observation 3 and (15) of Sec. 4.5.

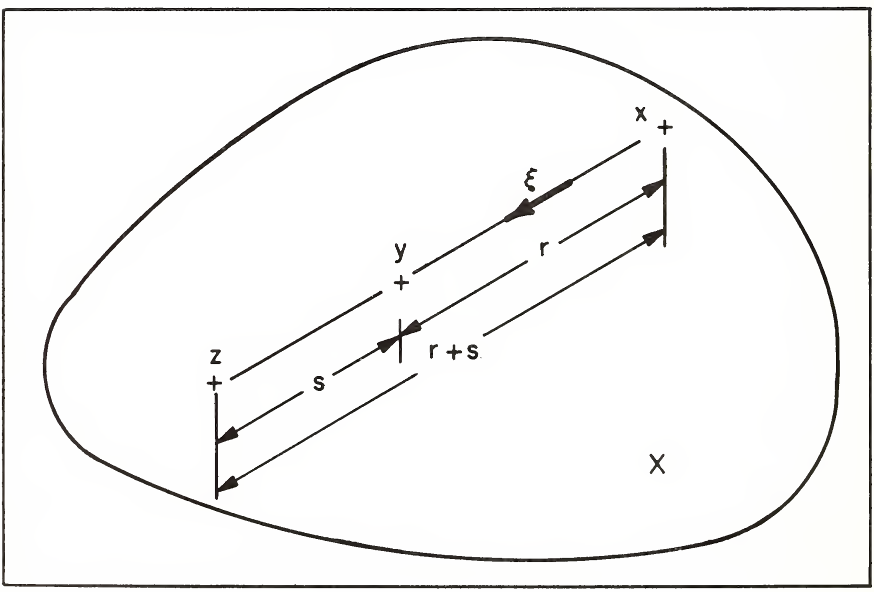


FIG. 9.7 The semigroup property for contrast transmittance.

Alternate Representations of Contrast Transmittance

In view of the fact that the component transmittances making up contrast transmittance (definition 3) are themselves expressible as ratios of radiances, it follows that the contrast transmittance of a path may also be represented as such. Indeed, from definition 3 and (4) and (5):

$$\boxed{\mathcal{T}_r(x,\xi) = \frac{N_r^o(z,\xi)}{N_r(z,\xi)} = T_r^o(x,\xi)\,\frac{N_o(x,\xi)}{N_r(z,\xi)}} \tag{19}$$

for every path $\mathcal{P}_r(x,\xi)$ with terminal point $z = x + r\xi$. Hence $\mathcal{T}_r$ may be usefully thought of as the ratio of the residual radiance to the apparent radiance of the associated path. The manner in which N_r is generated along $\mathcal{P}_r$ may be through either scattering or reflection or both. The latter mechanisms occur when $\mathcal{P}_r$ passes through interfaces, such as the air-water interface. Since $N_r(z,\xi)$ is decomposable into the sum of $N_r^o(z,\xi)$ and $N_r^*(z,\xi)$, ((5) of Sec. 3.13), (19) yields up still other useful forms, such as the following:

$$\mathcal{T}_r(x,\xi) = 1 - \frac{N_r^*(z,\xi)}{N_r(z,\xi)} \tag{20}$$

or:

$$\mathcal{T}_r(x,\xi) = \frac{1}{\left[1 + \dfrac{N_r^*(z,\xi)}{N_r^o(z,\xi)}\right]} \tag{21}$$

or:

$$\mathcal{T}_r(x,\xi) = \frac{N_r(z,\xi) - N_r^*(z,\xi)}{N_r^o(z,\xi) + N_r^*(z,\xi)} \tag{22}$$

or:

$$\mathcal{T}_r(x,\xi) = 1 - \frac{1}{\left[\dfrac{N_r^o(z,\xi)}{N_r^*(z,\xi)} + 1\right]} \tag{23}$$

or:

$$\mathcal{T}_r(x,\xi) = \frac{\left[\dfrac{N_r(z,\xi)}{N^*(z,\xi)} - 1\right]}{\left[\dfrac{N_r^o(z,\xi)}{N_r(z,\xi)} + 1\right]} \tag{24}$$

In virtue of the fact that radiance values are invariably nonnegative, and that decomposition property (5) of Sec. 3.13 holds, it follows that:

$$N_r^o(z,\xi) \leq N_r(z,\xi)$$

or: (25)

$$N_r^*(z,\xi) \leq N_r(z,\xi)$$

for every path $\mathcal{P}_r(x,\xi)$, so that (19) or any one of the preceding representations shows that:

$$\boxed{\mathcal{T}_r(x,\xi) \leq 1} \tag{26}$$

for every path $\mathcal{P}_r(x,\xi)$ in a general optical medium X. Some necessary and sufficient conditions that $\mathcal{T}_r(x,\xi)$ be 0 or 1 are clearly discernible from the preceding representations. Observe in particular that:

$$\mathcal{T}_r = T_r^o = T_r[-\alpha] \tag{27}$$

if $K = 0$ over $\mathcal{P}_r$, as is readily apparent from (14). Condition (27) holds for example when the path $\mathcal{P}_r$ is a horizontal uniformly lighted path in an extensive homogeneous portion of the sea or atmosphere. Furthermore, $\mathcal{T}_o = 1$ if and only if $N_o^* = 0$. It is clear that $N_o^* = 0$ if and only if the index of refraction over the path is continuous at the point of consideration of N_o^* (for an example where $\mathcal{T}_o \neq 1$, see (20) of Sec. 12.2). A contrast transmittance $\mathcal{T}_o$ which is distinct from 1 is called a *singular contrast transmittance*, and the associated path $\mathcal{P}_o$ a *singular path.*

Contrast Transmittance as an Apparent Optical Property

It was observed above (observation 2) that contrast transmittance is an apparent optical property, i.e., that it depends on the radiance field in the medium of interest. If the lighting along the path $\mathcal{P}_r$ is uniform, then we obtain such a corresponding extreme case as (27). On the other hand if the lighting along $\mathcal{P}_r$ is irregular, for example when $\mathcal{P}_r$ is directed through shadowed and sunlit regions, then the

values of $\mathcal{T}_r$ manifest these shadowings and lightings in a regular and predictable way, as the following two examples will show. To fix ideas we shall initially choose a very simple setting.

For the purposes of the first example, part (a) of Fig. 9.8 depicts a path $\mathcal{P}_r$ in a plane-parallel medium X(a,b) which is parallel to the boundaries of X(a,b) and at some depth y. The medium is stratified and has a stratified light field except in the shaded region shown, which simulates a shadowed part of the medium. For example, such a shadow may be produced by an isolated cloud over the ocean, or a ship shadow, etc. It is of interest to relate the contrast transmittance of $\mathcal{P}_r$ before and while it is shadowed. It is also of interest to compare this situation with that of

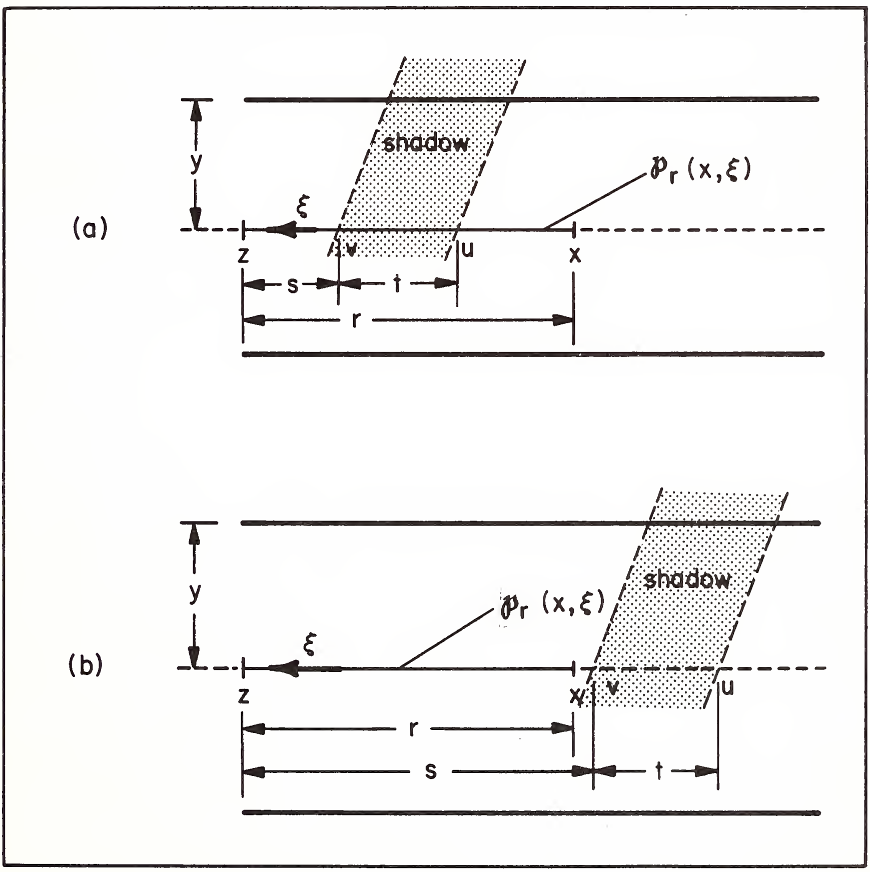


FIG. 9.8 The effect on the contrast transmittance of a path when shadows fall within and behind the path.

part (b) of the figure which depicts the shadow falling beyond the path $\mathcal{P}_r$.

It is at once clear from (19) that if "$N_r^{\#}(z,\xi)$" denotes the shadowed apparent radiance for $\mathcal{P}_r$, then since:

$$N_r^{\#}(z,\xi) \leq N_r(z,\xi) \quad ,$$

and $N^{\circ}(z,\xi)$ is unchanged, we have:

$$\boxed{\mathcal{T}_r^{\#}(z,\xi) \geq \mathcal{T}_r(z,\xi)} \quad (\mathcal{P}_r \text{ is shadowed internally}) \qquad (28)$$

for a shadowed path $\mathcal{P}_r$ as in (a) of Fig. 9.8. In the second example which is depicted, in case (b) of Fig. 9.8, both $N_r^{\circ}(z,\xi)$ and $N_r(z,\xi)$ are affected and exhibit a decrease. However $N_r^{\circ}(z,\xi)$ is decreased more than $N_r(z,\xi)$ so that:

$$\boxed{\mathcal{T}_r^{\#}(z,\xi) \leq \mathcal{T}_r(z,\xi)} \quad (\mathcal{P}_r \text{ has shadowed background}) \qquad (29)$$

for the path $\mathcal{P}_r$ shadowed as in (b) of Fig. 9.8. This may be seen by inspection of (20). For, by hypothesis, $N_r^*(z,\xi)$ is unaffected (no shadow from x to z) and $N_r(z,\xi)$ is decreased, so that the quotient in (20) is increased, the difference in (20) thereby decreased, which was the effect to be shown.

The apparent radiance formula (5) of Sec. 3.13 may be used to obtain quantitative estimates of the increase or decrease of $\mathcal{T}_r$ due to shadowing studied in the two preceding examples. To see how the formula is used, let us (for brevity) write: "N(z)" for $N(z,\xi)$ in (a) of Fig. 9.8 and "N(v)" for $N(v,\xi)$, and so on. since ξ is fixed all along the path. Further "N_s^*" will briefly denote $N_s^*(z,\xi)$, and "T_s°" will be short for $T_s^{\circ}(v,\xi)$. Then for the portion of path $\mathcal{P}_r$ in (a) of Fig. 9.8 extending from v to z , we have from (5) of Sec. 3.13:

$$N(z) = N(v)T_s^{\circ} + N_s^*$$

Further, for the segment from u to v : before the path is shadowed:

$$N(v) = N(u)T_t^{\circ} + N_t^*$$

where we have written "N(u)" for $N(u,\xi)$. If the path is now shadowed, then N_t^* decreases to, say $N_t^{\#}$. Let us write (ad hoc):

$$\text{"c"} \quad \text{for} \quad N_t^{\#}/N_t^* \quad .$$

We can now estimate the magnitude of the shadowed apparent radiance $N^{\#}(z)$ of $\mathcal{P}_r$ as follows: Let "$N^{\#}(v)$" denote the shadowed apparent radiance at v. We then have:

$$
\begin{aligned}
N^{\#}(v) &= N(u)T_t^o + N_t^{\#} \\
&= N(u)T_t^o + cN_t^* \\
&= N(u)T_t^o + N_t^* + (c-1)N_t^* \\
&= N(v) + (c-1)N_t^*
\end{aligned}
$$

Returning to $N^{\#}(z)$, we have:

$$
\begin{aligned}
N^{\#}(z) &= N^{\#}(v)T_s^o + N_s^* \\
&= [N(v) + (c-1)N_t^*]\, T_s^o + N_s^* \\
&= N(z) + (c-1)T_s^o N_t^*
\end{aligned}
$$

Hence the desired representation of $N(z) - N^{\#}(z)$ is"

$$
N(z) - N^{\#}(z) = (1-c)T_s^o N_t^* \tag{30}
$$

Now in general, the smaller t is, the smaller N_t^* will be and so by (30), the less effect the shadow will have. Further, the farther away the shadow is (i.e., the greater s is) from the point of observation, the less effect the shadow will have, since T_s^o decreases as s increases in real media. A similar analysis can be made for part (b) of Fig. 9.8, and it is left to the reader to show that, in this case:

$$
N(z) - N^{\#}(z) = (1-c)T_s^o N_t^* \tag{31}
$$

From (30) and (31) we see that both cases are covered by the same type of formula. It is interesting to observe that if the shadow region from u to v in Fig. 9.8 straddles point x in just the right way, the contrast transmittance of the path $\mathcal{P}_r(x,\xi)$ is unchanged, even though, as (30) and (31) indicate, there are definite changes in the lighting conditions.

An examination of the preceding arguments would show that no essential use was made of the plane-parallel structure of the medium depicted in Fig. 9.8, nor of its stratified light field. Furthermore, the shadowing factor c may just as well have been a lighting factor (i.e., $c > 1$), without affecting the algebraic structure of the resulting equations (30) and (31). We use these observations to generalize the results and to summarize our findings as follows.

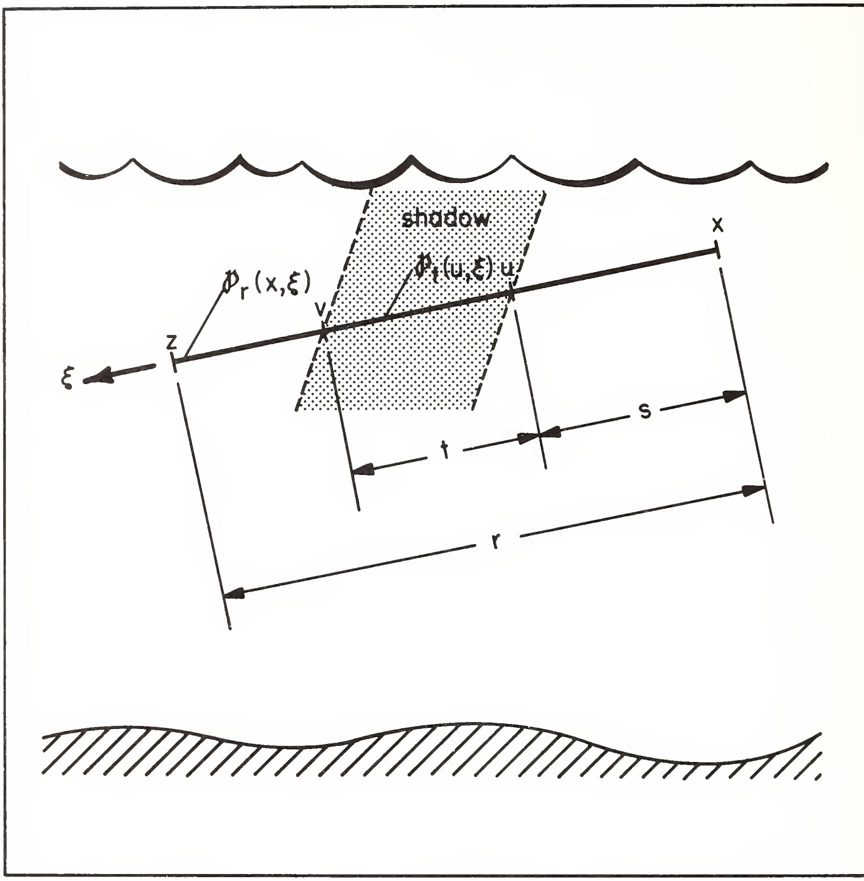


FIG. 9.9 The general setting for Fig. 9.8.

Let $\mathcal{P}_r(x,\xi)$ be a path in a general optical medium X, and let $\mathcal{P}_t(u,\xi)$ be a *proper collinear path* with respect to $\mathcal{P}_r(x,\xi)$ i.e., for some positive or negative number s, $u = x + s\xi$ and $r \geq s + t$ (see Fig. 9.9 in which distance is mreasured positive from x to z, and negative from z to x). Suppose that the path radiance $N_t^*(v,\xi)$ of $\mathcal{P}_t(u,\xi)$ (where $v = u + t\xi$) changes by a factor c. This change generally affects the apparent radiance $N_r(z,\xi)$, where $z = x + r\xi$. Let "$N_r^\#(z,\xi)$" denote this new apparent radiance, then:

$$\boxed{N_r(z,\xi) - N_r^*(z,\xi) = (1-c)T^o_{r-(s+t)}(v,\xi)N_t^*(v,\xi)} \quad (32)$$

If the proper collinear path $\mathcal{P}_t(u,\xi)$ is wholly contained within $\mathcal{P}_r(x,\xi)$ (i.e., $s \geq 0$) then the contrast transmittance

$\mathcal{T}_r^{\#}(x,\xi)$ corresponding to the factor c is greater or less than $\mathcal{T}_r(x,\xi)$ according as c is $\gtrless 1$. On the other hand, if the proper collinear path $\mathcal{P}_t(u,\xi)$ is behind and disjoint from $\mathcal{P}_r(x,\xi)$ (i.e., $s < 0$, and $s + t < 0$) then the contrast transmittance $\mathcal{T}_r^{\#}(x,\xi)$ corresponding to the factor c is less or greater than $\mathcal{T}_r(x,\xi)$ according as c is $\gtrless 1$. A quantitative estimate of $\mathcal{T}_r^{\#}(x,\xi)$ may be based on (32) by solving for $N_r^{\#}(z,\xi)$ and using any one of (19)-(24). In case $s \leq 0$, then $N_r^{o}(z,\xi)$ is also changed, and must be computed accordingly, using (32), and (5) of Sec. 3.13. In general three cases should be distinguished for $\mathcal{T}_r^{\#}(x,\xi)$: the proper collinear path $\mathcal{P}_t(u,\xi)$ is either (a) contained in $\mathcal{P}_r(x,\xi)$, (b) behind and disjoint from $\mathcal{P}_r(x,\xi)$ or, (c) behind and overlapping $\mathcal{P}_r(x,\xi)$, i.e., $s < 0$ and $s + t > 0$.

On the Multiplicity of Apparent Radiance Representations

One final observation may be made at this point on the use of (5) of Sec. 3.13 in deducing the preceding properties of contrast transmittance, and which helps cast some light on an interesting general fact about the concept of apparent radiance. The attentive reader will have noticed that there appears to be an infinite number of choices in the manner of representing a radiance $N(x,\xi)$ as an apparent radiance associated with some path $\mathcal{P}_r(x,\xi)$. This fact was essentially observed in Sec. 3.13 in the discussion of (5) of that section. However, the present applications of that equation in deducing (30), (31), and (32) bring home the multiplicity of the representations with renewed force; and we shall now formalize this fact for future reference. The observation may be phrased generally as follows. Suppose $N(z,\xi)$ is the radiance along the direction ξ at some point z in an optical medium X. If now we place a path (an imaginary construct) in X with direction ξ so that x is its initial point and $z = x + r\xi$ is the terminal point (Fig. 9.7), then we may immediately reorient our conception of $N(z,\xi)$ from that of a primitive radiance in X to that of an apparent radiance of the medium at x associated with the path $\mathcal{P}_r(x,\xi)$. That is, by (5) of Sec. 3.13, we can write:

$$N(z,\xi) = N(x,\xi)T_r^{o}(x,\xi) + N_r^{*}(z,\xi) \tag{33}$$

and to point up the fact that $N(z,\xi)$ and $N(x,\xi)$ are viewed in the framework of the path $\mathcal{P}_r(x,\xi)$, we may, as is customary, attach "r" and "o" subscripts to them, respectively. By imagining still another path $\mathcal{P}_s(y,\xi)$ with direction ξ and initial and terminal points y and $z = y + s\xi$, we can represent the same $N(z,\xi)$ in (33) as:

$$N(z,\xi) = N(y,\xi)T_s^{o}(y,\xi) + N_s^{*}(z,\xi) \tag{34}$$

This phenomenon of the multiplicity of possible representations of a given measurable radiance $N(z,\xi)$ with respect to

imaginary paths in an optical medium is reminiscent of, and actually logically related to, the freedom of choice of the parameters x and z (Holding y fixed) in the statements of the principles of invariance (e.g., in Example 3 of Sec. 3.7) or the invariant imbedding relation (e.g., in Example 4 of Sec. 3.7).

9.6 Classification of Optical Properties

We conclude this chapter with a summary of the main optical properties introduced and developed in the present work. We shall classify the properties in several ways, according to the dimension of the medium to which they primarily apply, and according to whether they are local, global, inherent or apparent optical properties.

TABLE 1

Generic Inherent Optical Properties for One-, Two-, and Three-Dimensional Media

Dimension	Optical Property	Section
1 (paths)	$\mathbf{R}$, $\mathbf{T}$	3.17
2 (surfaces)	$r_{\pm}$, $t_{\pm}$	3.3
3 (solids)	$\mathcal{S}$	3.8

The term "generic," used in Table 1 to describe the listed optical properties, refers to their ability to generate all the secondary optical properties associated with the respective media, as explained in the various sections of Chapter 3. Thus, e.g., $\mathcal{S}(X;a,b)$ can generate all the standard reflectance and transmittance operators for plane-parallel media.

Below are tables of optical properties for plane-parallel (or generally one-parameter) optical media, the media of principal interest in hydrologic optics. The properties above each level may be used to deduce those on that level in the manner explained in the notes or references accompanying each table.

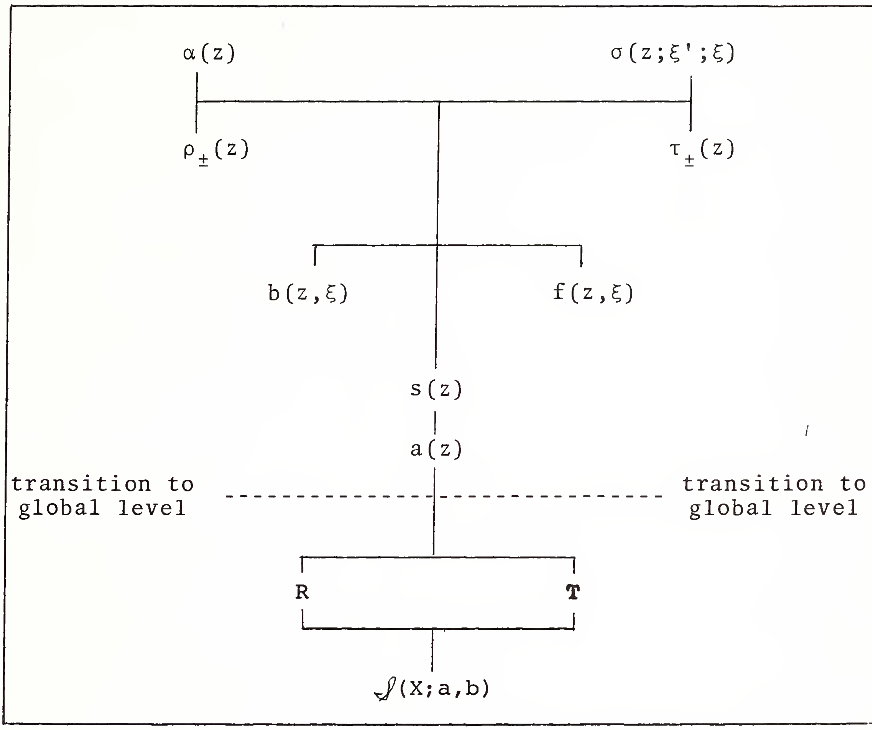
Several explanatory comments on Table 2 can be made. First, the operators $\rho_{\pm}(z)$ and $\tau_{\pm}(z)$ were defined in (3) and (4) of Sec. 7.1. The functions $f(z,\xi)$ and $b(z,\xi)$ are added to complement $f(z,\pm)$, $b(z,\pm)$ of Table 4 below, and are defined by writing:

$$\text{"}f(z,\xi)\text{"} \quad \text{for} \quad \int_{\Xi_+(\xi)} \sigma(z;\xi;\xi')\,d\Omega(\xi') \tag{1}$$

$$\text{"}b(z,\xi)\text{"} \quad \text{for} \quad \int_{\Xi_-(\xi)} \sigma(z;\xi;\xi')\,d\Omega(\xi') \tag{2}$$

TABLE 2

Local Inherent Optical Properties for Plane-Parallel Media



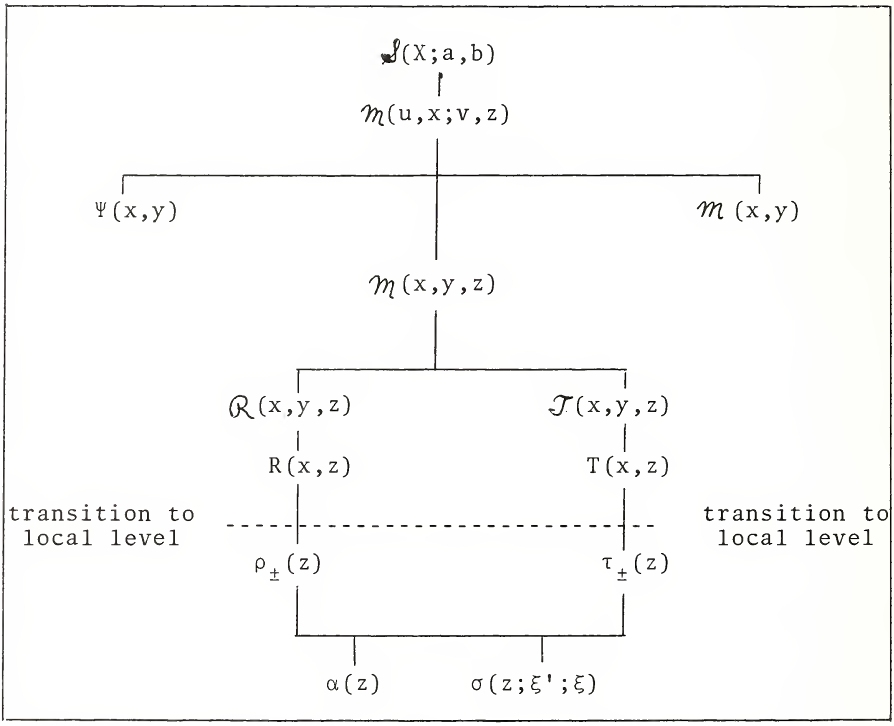
Here $\Xi_\pm(\xi)$ is the set of all directions ξ' such that $\xi \cdot \xi' \geq 0$ (for +) or $\xi \cdot \xi' < 0$ (for −). If the medium is isotropic, then $f(z,\xi)$ and $b(z,\xi)$ are independent of ξ.

The dashed line in the diagram of Table 2 represents the end of Table 2 and serves first of all to indicate the transition to the global level of Table 3, and is also included in order to show how Tables 2 and 3 are related, and finally to emphasize the fact that the pair (α, σ) is fundamental in the sense of Definition 3 of Sec. 9.1. The proof of this feature of (α, σ) is given in Sec. 22 and Sec. 23 of Ref [251]. The operators **R** and **T** are given in (5) and (12) of Sec. 3.17, and $\mathcal{J}$ is introduced in (6) of Sec. 3.8 and studied in Example 4 of Sec. 3.9.

Table 3 performs a double task of describing both the inherent and apparent global properties of plane-parallel media. The inherent properties are in force in Table 3 when radiance is being used; the apparent properties are in force when irradiance is being used. The operators

TABLE 3

Global Inherent or Apparent Optical Properties
for Plane-Parallel Media



$\mathcal{M}(u,x;v,z)$ and $\mathcal{M}(x,y)$ are introduced in Examples 6 and 7 of Sec. 3.7. The operator $\Psi(x,y)$ is introduced in Example 3 of Sec. 3.9. The invariant imbedding operator $\mathcal{M}(x,y,z)$ is introduced in Example 4 of Sec. 3.7, along with its component operators $\mathcal{R}(x,y,z)$ and $\mathcal{T}(x,y,z)$. The standard operators $R(x,z)$ and $T(x,z)$ are introduced in Sec. 3.6. The transition to the local level is indicated by a dashed line which signals the end of Table 3 and is also included in the table to emphasize the fact that $\mathcal{S}(X;a,b)$ is a fundamental optical property (in the sense of Definition 3, Sec. 9.1). The proof that $\mathcal{S}(X;a,b)$ is a fundamental optical property is based on the derivations given in Sec. 125 of Ref. [251].

The optical properties of Table 4 are written for undecomposed irradiance fields. By appending star or circle superscripts,* the properties for diffuse or residual

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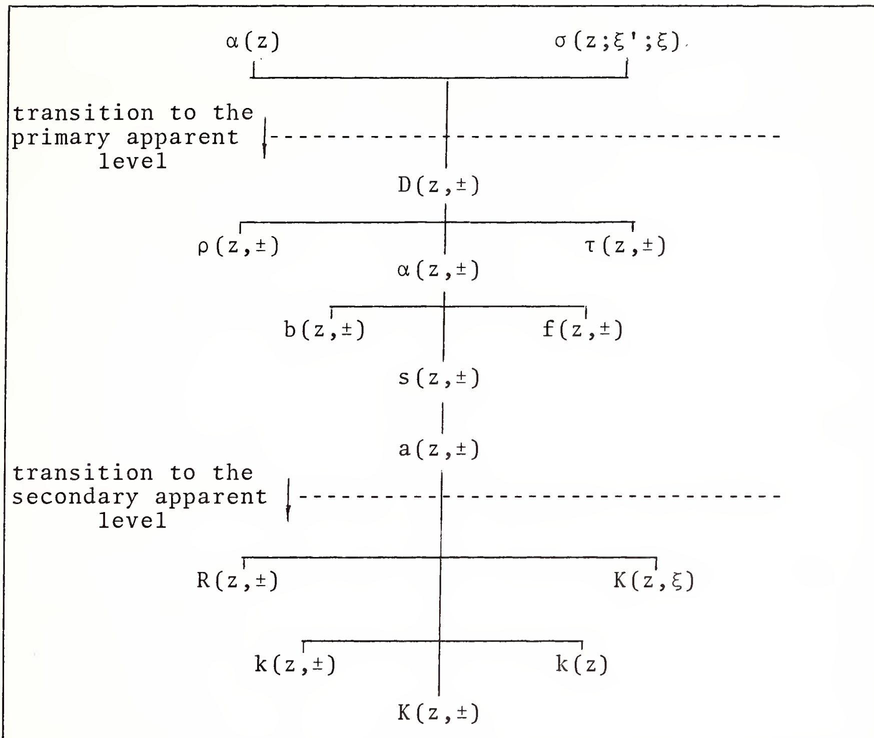
*The basis for this notational device is described in Sec. 8.4.

TABLE 4

Local Apparent Optical Properties for
Plane-Parallel Media



irradiance are obtained. In the interests of brevity, these additional concepts are not diagrammed. The primary level of optical properties in Table 4 for the most part parallel their inherent counterparts in Table 2. They may be thought of as hybrids resulting from the union of inherent optical properties and the light field. The secondary apparent level summarizes some optical properties which are on the borderline between local and global properties. For example, $R(z\ -)$ is the reflectance at level z, and thus, being defined at a point at level z, is ostensibly local in nature. However, the value of $R(z,-)$ is intimately tied to the values of the light field and the inherent optical properties of the medium at all levels above and below level z. The classical counterparts to the secondary optical properties of Table 4 arose in the solutions of the one-D two-flow models of the light field. The secondary properties appearing in Table 4 are the exact, directly observable counterparts to the earlier one-D model concepts. The various primary properties occurring in Table 4 are defined throughout Chapter 8, the secondary properties (including $K(z,\xi)$) are defined in Sec. 9.2.

9.7 Bibliographic Notes for Chapter 9

The development of the directly observable quantities for light fields in natural hydrosols, as presented in Sec. 9.2, is based on Ref. [222]. A published version of this reference was made available in the literature in Ref. [247]. The covariation of $K(z,-)$ with $D(z,-)$ is based on the results in Ref. [242]. The analytical representation of the observable reflectance function developed in Sec. 9.4 is drawn from Ref. [246].

The concept of contrast in the form of a relative difference of two radiances occurs in the writings of Mecke [174] and was applied by Koschmieder [141] in his classical studies of 1924. Systematic use of the concept of contrast was made by Duntley [71,72] in 1948 in the study of visibility in the atmosphere. Further uses of contrast in the atmosphere are given in Middleton's [177] work. Applications of the contrast concept to hydrologic optics were reported [82] and systematically generalized [210]. These generalizations were subsequently applied to the atmospheric context in [80] and in [228]. The discussion of Sec. 9.5 is an outgrowth of the work in [210].

The optical properties of plane-parallel media were first studied via the one-D model (the Schuster two-flow theory) discussed in Sec. 8.10. The definitions and classifications of the optical properties in Sec. 9.1 and Sec. 9.6 are based on the unifying concept of the interaction principle and are for the most part new. Some preliminary classifications preparing the way to those in Sec. 9.1 and 9.6 are given in [210] and [247].

CHAPTER 10

OPTICAL PROPERTIES AT EXTREME DEPTHS

10.0 Introduction

In this chapter we examine some theoretical and experimental evidence for possible regular behavior of apparent optical properties at deep and shallow depths in natural bodies of water. These extreme depths in a natural hydrosol are the settings of interesting and complex radiometric interactions of light from the sky with the body of the medium. The observed interactions at these depths are exaggerated either because of the extreme proximity of the air-water boundary or because of its extreme remoteness and thus present to both theoretical and experimental workers a challenging puzzle to unravel and bring conceptual order to the understanding of the light field observations at these depths. In the attempt to understand the radiometric phenomena at extreme detphs, we shall be led to supplement the collection of laws governing the optical properties derived in Chapter 9 (which hold for all depths in a natural hydrosol). Furthermore, by concentrating on the extreme depths in the present chapter, we can extract correspondingly more detailed behavior of the observable K-functions of both the radiance and irradiance fields. Our investigations are based on the general equation of transfer and on the exact two-flow equations for irradiance of Chapter 8; in particular we shall make extensive use of the accurate two-D model for irradiance fields developed in Chapter 8.

We shall begin with the study of irradiance fields at shallow depths in media with calm surfaces, our primary aim being to discern from both theoretical and experimental clues, the precise nature of the depth-behavior of the upward and downward irradiance fields $H(z,\pm)$ at these depths. By examining these fields via their K-functions $K(z,\pm)$ we shall, as it were, be examining them under a powerful magnifying glass by means of which every tendency of decay or growth of $H(z,\pm)$ is limned in bold relief against a conceptual background of algebraic signs and magnitudes of derivatives.

The second half of the chapter is devoted to the study of the natural light fields at great optical depths as they occur in oceans, harbors, and deep lakes. At remote distances from the generally disturbing air-water boundary of the medium, the radiance distributions eventually attain a smooth, characteristic shape independent of the external lighting conditions and dependent only on the inherent optical properties α , σ of the medium. The problem of the nature of the

light field at great depths in natural waters has been completely solved only recently. As a result of having a theoretical basis for the existence of the regular behavior of the light field at great depths (for most practical purposes, beyond 10-20 attenuation lengths), we can justify certain simplifications of the models for light fields below such depth intervals. In particular, in such regions the classical canonical form of the equation of transfer (Chapter 4) for radiance, and the two-D model for irradiance (Chapter 8) can be demonstrated to serve as accurate tools with which to quantitatively predict the magnitudes of the natural light fields.

10.1 On the Structure of the Light Field at Shallow Depths: Introductory Discussion

In this section experimental determinations of the upwelling and downwelling irradiances are studied with the purpose of explaining certain observed regular nonlinear trends in the semilog plots of these irradiances, principally at shallow depths in media with flat calm surfaces. We shall develop a mathematical model from the general equation of transfer which describes these irradiances in great detail over the shallow-depth range. The model explains the observed phenomena in terms of the inherent optical properties of the medium and its external lighting conditions. On the basis of experimental evidence, cited below, and on the basis of supporting theory, the following hypothesis about light fields in all homogeneous natural hydrosols is proposed: (a) The ratio of the upwelling irradiance to the downwelling irradiance, i.e., the observable reflectance function $R(z,-)$, is invariably monotonic increasing or decreasing at shallow depths with increasing depth (depending on the medium) and approaches a limit which is independent of the external lighting conditions and which depends only on the inherent optical properties of the medium. (b) The logarithmic derivatives, i.e., the K-functions $K(z,\pm)$, of the upwelling and downwelling irradiance at shallow depths are monotonic increasing or decreasing with increasing depth (depending on the medium) and approach a common limit which is independent of the external lighting conditions and which depends only on the inherent optical properties of the medium. In this way we arrive at a fairly detailed understanding of the light field at extreme depths (shallow and deep) in all homogeneous natural hydrosols.

To set the stage for the general reader, the following observations on the empirical roles of the K and R functions will be helpful. For many practical purposes in applied hydrologic optics the downwelling irradiance $H(z,-)$ at a depth z in a natural hydrosol can be represented by the following simple formula

$$H(z,-) = H(0,-)e^{-Kz} \quad , \tag{1}$$

where K is a fixed number which characterizes the overall flux transmitting properties of the hydrosol. A similar formula may be used to determine the upwelling irradiance $H(z,+)$ at any depth z :

$$H(z,+) = H(0,+)e^{-Kz} \quad , \tag{2}$$

where--again for many practical purposes--K is a fixed number and in fact identical to the one appearing in (1). Simple models of the light field, such as the two-D and one-D models of Secs. 8.5 and 8.6 supply detailed formulas of the kind (1) and (2).

Still another practical formula is the one which describes the depth dependence of scalar irradiance $h(z)$ at each depth z:

$$h(z) = h(0)e^{-Kz} \quad , \tag{3}$$

where K is the same number as that appearing in (1) and (2).

In practice $H(z,+)$ and $H(z,-)$ are measured by suitably designed horizontal flat plate collectors exposed to the appropriate hemisphere, and $h(z)$ is measured by a suitably designed spherical collector. In view of (1), (2), and (3), relatively quick estimates of a K for a particular natural body of water can be obtained by measuring any one of these three radiometric quantities at two distinct depths, and using the formula:

$$K = \frac{1}{(z_2 - z_1)} \ln \frac{A(z_1)}{A(z_2)} \quad , \tag{4}$$

where "$A(z)$" stands for any one of the three quantities: $H(z,+)$, $H(z,-)$, or $h(z)$ at depth z.

The practical procedures of hydrologic optics, summarized in formulas (1)-(4), are quite analogous to the following well-known procedure used in applied heat conduction studies to estimate the temperature $T(t)$ of a cooling spherical body at time t immersed in an infinite bath of zero temperature:

$$T(t) = T(0)e^{-kt} \quad , \tag{5}$$

where k is a known fixed number which characterizes the overall heat conducting properties of the material comprising the spherical body. Conversely, (5) may be used to estimate k by measuring $T(t)$ at two distinct times and using a formula exactly analogous to (4).

The specialists who use (5) are aware of the fact that it is a useful approximate formula which becomes an exact formula for $T(t)$ in the limit as $t \to \infty$. They also realize that (5) becomes quite inadequate for relatively accurate estimates of $T(t)$ whenever t is small, and must resort in such estimates to more general forms representing $T(t)$. These more general forms, of which (5) is a special limiting case, are well known and are solidly founded in the general

theory of heat conduction, and experimental fact, and may be found in standard treatises on heat conduction.

Equations (1)-(3) are regarded by the specialists in hydrologic optics in much the same way as (5) is regarded in its own discipline: They are useful approximate relations which can be shown to become exact formulas for $H(z,\pm)$ and $h(z)$ in the limit as $z \to \infty$ in deep homogeneous plane parallel optical media (see, e.g., Sec. 10.7). Perhaps what is not well known--or at any rate not fully realized--is that, like (5) these equations do not exactly represent $H(z,\pm)$ or $h(z)$ for small values of z, even in homogeneous hydrosols with uniform external lighting conditions and perfectly calm air-water surfaces. Thus, for relatively accurate estimates of $H(z,\pm)$ and $h(z)$ such as those required in basic scientific studies of the light fields in natural hydrosols, (1)-(3) are quite inadequate. They do not represent the small but experimentally demonstrable departures from linearity of the semilog plots of $H(z,\pm)$ and $h(z)$.

What is required at present in the discipline of hydrologic optics is a set of more general formulas which can accurately represent the quantities $H(z,\pm)$ and $h(z)$ in the small z ranges and which reduce to these simpler formulas in the limit as $z \to \infty$.

One of the two purposes of the following sections is to present a set of formulas for $H(z,\pm)$ which yield a closer approximation to reality than (1) and (2). The search for these formulas was motivated by the results of recently performed accurate measurements in the light field in real natural hydrosols, and their derivations are founded on the tenets of general radiative transfer theory.

The second purpose of the present discussion is to examine the resulting formulas for indications of possible general qualitative rules that may be hypothesized about the fine structure of shallow-depth light fields, and to put the hypotheses into forms which will be amenable to further theoretical study or experimental verification. On the basis of the models constructed below it was possible to formulate three such hypotheses about the quantities:

$$K(z,\pm) = -\frac{1}{H(z,\pm)}\,\frac{dH(z,\pm)}{dz} \tag{6}$$

and

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad , \tag{7}$$

introduced in Chapter 9. These hypotheses governing the functions $K(z,\pm)$ and $R(z,-)$ are presented in detail below in Sec. 10.3, but for the present we shall undertake some preliminary discussion of the roles of these functions in the study of natural light fields.

As noted in Sec. 9.3, the quantities $K(z,+)$ and $K(z,-)$ are simply the slopes of the semilog plots of $H(z,+)$ and $H(z,-)$. According to the simple formulas (1) and (2), these slopes do not change with depth and in fact are of the form:

$$K(z,+) = K(z,-) = K \quad ,$$

where K is defined in (1) and (2). Careful experiments show, however, that $K(z,+)$ and $K(z,-)$ are generally distinct numbers that do change with depth. Furthermore, we shall see in Sec. 10.7 that, in homogeneous media,

$$\lim_{z\to\infty} K(z,+) = \lim_{z\to\infty} K(z,-) = k_\infty$$

where k_∞ is a number which depends only on the inherent optical properties of the medium and is completely independent of the external lighting conditions on the upper boundary of the medium. One of the goals of the present chapter is to find out something about the nonlinear behavior of $K(z,\pm)$ at relatively *small* depths.

The quantity $R(z,-)$ summarizes the flux transmitting and reflecting properties of the medium both above and below the hypothetical plane at depth z. According to the simple formulas (1) and (2),

$$R(z,-) = \frac{H(0,+)}{H(0,-)} \quad ,$$

a fixed number for all z. Careful experiments show, however, that $R(z,-)$ changes with depth; and in all homogeneous media it will be shown (Sec. 10.7), to approach a well-defined limit as $z \to \infty$:

$$\lim_{z\to\infty} R(z,-) = R_\infty \quad ,$$

where R_∞ is a number which depends only on the inherent optical properties of the hydrosol and is completely independent of the external lighting conditions on the upper boundary of the medium. Another of our present goals is to understand the nonlinear behavior of $R(z,-)$ for relatively small values of z.

10.2 Experimental Basis for the Shallow Depth Theory

To prepare the groundwork leading to the theory of the light field at shallow depths, we now consider some experimental data which supplies graphic evidence of the nonlinear depth behavior of $K(z,\pm)$ and $R(z,-)$ in near-surface regions of a specific hydrosol. The experimental evidence presented in this and the following sections has been computed from the data obtained in Lake Pend Oreille, Idaho, by J. E. Tyler [298].

Figure 10.1 depicts the semilog plots of $H(z,+)$ and $H(z,-)$ over the range of depths $5 \le z \le 55$ meters; $H(z,\pm)$ are associated with a wavelength interval of width 64 mμ centered at 480 mμ. This depth range corresponds to a range of about 20 optical depths, so that the light field in the vicinity of 50 meters should have for all practical purposes attained the asymptotic limit--assuming complete homogeneity of the medium.

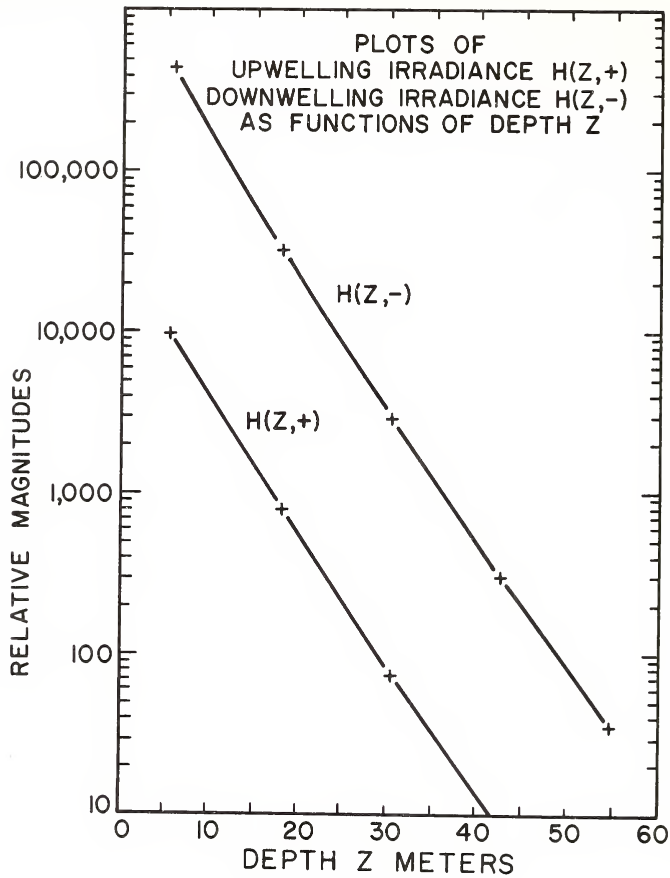


FIG. 10.1 Irradiance plots (480 mμ) from Tyler's study of Lake Pend Oreille, Idaho, spring 1957.

Just how close is the present hydrosol to being homogeneous? To answer this, we must know the values of the volume attenuation function α within the medium. Figure 10.2 shows a plot of α versus depth for the present medium. An optical medium is by definition *homogeneous* if α is a constant function within that medium. It is clear from the α plot of Figure 10.2 that the hydrosol was not strictly homogeneous at the time of the experiment. There is about a 12 percent variation in the values of α over the indicated depth range. In several places, in particular the bracketed range, there is a relatively abrupt change of the order of 5 percent in the values of α. For many practical purposes these changes are negligible. However, for the specific purposes of the present discussion, these changes are of extreme interest. In addition to $\alpha(z)$, the values $a(z)$ of the volume absorption function are plotted for several depths.

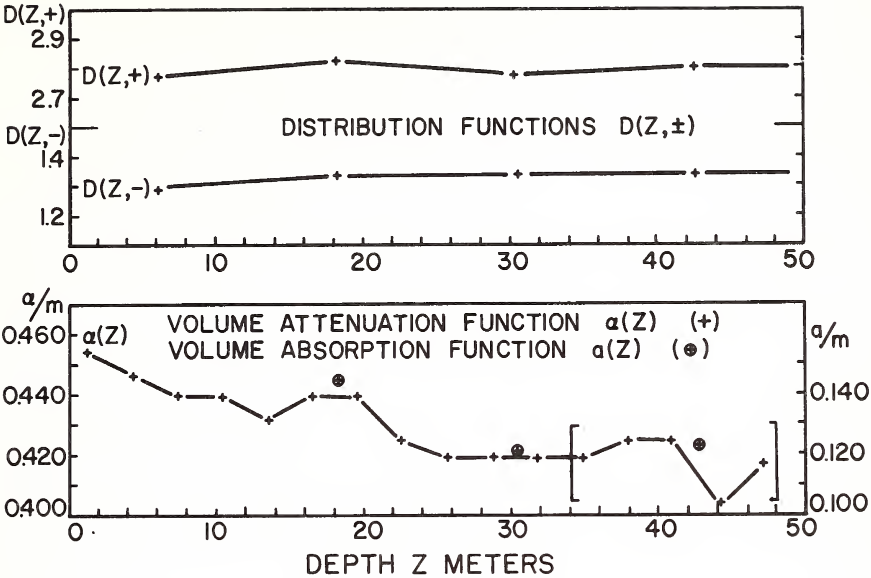


FIG. 10.2 Optical properties $\alpha(z)$, $a(z)$, and $D(z,\pm)$ of Lake Pend Oreille, Idaho (for 480 mμ) as determined by Tyler in spring 1975.

Observe how tne values $a(z)$ tend to follow the changes in $\alpha(z)$. This feature will be noted again later in this study when the mathematical model is being discussed. With this background information in mind we may now turn to a detailed examination of the plots of $H(z,\pm)$.

Observe that each of the plots in Fig. 10.1 exhibits a small but noticeable nonlinearity. The curves are slightly concave upward, indicating relatively steep slope at shallow depths and less steep slopes at greater depths. To facilitate the examination of the logarithmic slopes, they have been plotted as functions of depth in Fig. 10.3. Both $K(z,+)$ and $K(z,-)$ exhibit a uniform downward trend toward a common asymptote defined by the horizontal line across the figure at ordinate $k_\infty = 0.178$/meter. This value was obtained using the fact, cited earlier, that $K(z,\pm)$ has a common limit and then performing a suitable extrapolation based on this fact (see Ref. [263]).

The uniform approach of $K(z,\pm)$ to this common limit appears to be interrupted in the neighborhood of 40 meters. There appears to be some optical disturbance in the medium within the immediate depth range (bracketed in Fig. 10.3) that results in a marked deviation of these curves from their expected paths. We can explain this anomalous behavior on the basis of our observations of the depth dependence of $\alpha(z)$ in Fig. 10.2. The abrupt change in the values of α in the same depth range appear to hold the key to the explanation when the following formulas are examined:

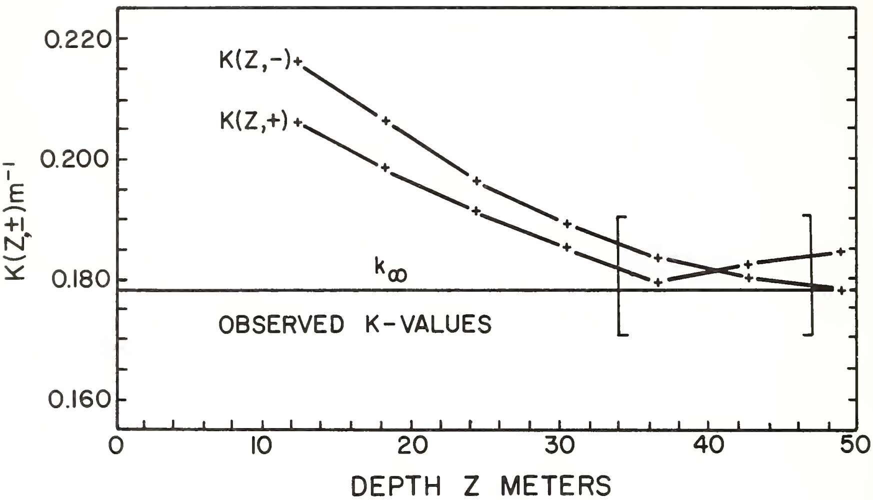


FIG. 10.3 K-function values (at 480 mμ) for Lake Pend Oreille, Idaho, spring 1957, computed from Tyler's data.

$$K(z,-) = \alpha(z,-) - \frac{\int_{\Xi_-} N_*(z,\xi)\,d\Omega(\xi)}{H(z,-)} , \qquad (1)$$

$$K(z,+) = -\alpha(z,+) + \frac{\int_{\Xi_+} N_*(z,\xi)\,d\Omega(\xi)}{H(z,+)} . \qquad (2)$$

These are exact formulas relating $K(z,\pm)$ to the values of α and the angular distribution of the light field at depth z. These formulas need not be derived here; they readily follow from the representations of the K-functions in (18) and (19) of Sec. 9.2. See the derivation of (26) of Sec. 9.2. Moreover, the definitions of the quantities $\alpha(z,\pm)$:

$$\alpha(z,\pm) = \alpha(z)D(z,\pm)$$

are given in Sec. 8.3. Here $D(z,\pm)$ are the values of the distribution functions for the downwelling (-) and upwelling (+) streams of radiant flux at depth z. Figure 10.2 shows a plot of $D(z,+)$ and $D(z,-)$ for the present hydrosol. It is evident that these quantities are nearly constant over the entire depth range under study.

On the basis of (1), whenever there is an abrupt change in the values $\alpha(z)$ over some small depth interval and whenever $D(z,-)$ is relatively fixed over this depth interval (so that the integral term is relatively constant), we predict that $K(z,-)$ must exhibit a change in the *same direction* as that of $\alpha(z)$. Thus if $\alpha(z)$ abruptly decreases, $K(z,-)$ is expected to exhibit a decrease in value.

On the basis of (2), on the other hand, under the same conditions, we should expect a simultaneous change of $K(z,+)$, but in the *opposite direction* as that of $\alpha(z)$. Thus if $\alpha(z)$ abruptly decreases, $K(z,+)$ is expected to exhibit an increase in value.

Returning now to Figs. 10.3 and 10.2, these predictions are apparently borne out by the portions of the α, and K curves in the bracketed depth range. Therefore the abrupt inhomogeneity in the structure of the hydrosol in this depth range appears to induce the observed interruption of the orderly trend of the K - curves toward their limit.

One might inquire why the comparable change in α in the vicinity of 20 meters does not produce a similar marked effect on the K - curves. The answer lies in the fact that the light field in the vicinity of 5-30 meters (2-12 attenuation lengths) is evidently still in the process of settling down and attaining a spatial steady state configuration, so that changes in K - values are naturally relatively great in this region; changes in α - values thus have relatively little additional influence on the fine structure of the K - functions, and their effects are obscured by the settling-down changes taking place. However, at around 40 meters (about 16 attenuation lengths) the light field has begun to assume its asymptotic angular structure. Any abrupt change in $\alpha(z)$ would now cause the entire smoothing process to recommence; in particular the K - values are now very close to their limit, and the effect of any inhomogeneity would be relatively magnified.

As a final step in the examination of the experimental evidence, we turn to Fig. 10.4 which exhibits a plot of $R(z,-)$ versus depth. In this case there is a uniform upward trend, as depth increases, of the values of $R(z,-)$ toward the limiting value $R_\infty = 0.0278$. This limiting value may be found from the formula:

$$R_\infty = \frac{k_\infty - a(-)}{k_\infty + a(+)} \tag{3}$$

where

$$a(\pm) = aD(\pm) \quad .$$

The quantities $D(\pm)$ are the limits of $D(z,\pm)$ as $z \to \infty$, and are found from the plots of $D(z,\pm)$ in Fig. 10.2. The values used were $D(+) = 2.77$, $D(-) = 1.33$; k_∞ is as defined in Fig. 10.3 and a was taken as the value of the volume absorption function at depth 42 meters: $a(42) = 0.123$ per meter. The basis for (3) will be established in Sec. 10.7. However,

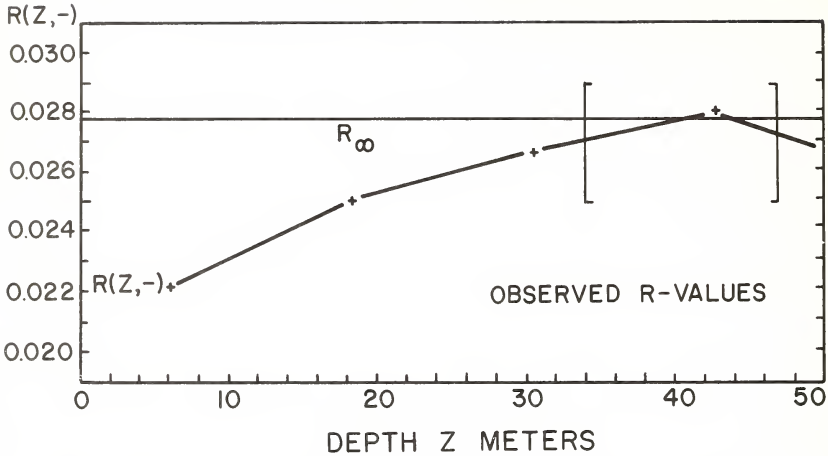


FIG. 10.4 Reflectance function data (at 480 mμ) for Lake Pend Oreille, Idaho, spring 1957, computed from Tyler's data.

we already have a version of it given us by the two-D model for irradiances in (102) of Sec. 8.7.

As in the case of the plots of $K(z,\pm)$, the plot of $R(z,-)$ exhibits a change in trend in the bracketed depth range discussed earlier. From the exact representation:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)}$$

of the function $R(z,-)$ (given in (25) of Sec. 9.2), and the observed changes in $K(z,+)$ and $K(z,-)$, we see that the observed anomaly, namely the downward trend exhibited by $R(z,-)$ in the vicinity of 40 meters, is traceable directly to the abrupt change of $\alpha(z)$ in this vicinity. Another way to see the cause of the change of slope of $R(z,-)$ in Fig. 10.4 is to examine (32) of Sec. 9.2 in the neighborhood of the crossing of the K-curves in Fig. 10.3.

Summary of the Experimental Evidence

We may now summarize the preceding observations:

(a) Over the depth ranges where the hydrosol is practically homogeneous, the magnitudes of the K-functions exhibit a *monotonic* decrease with increasing depth, with $K(z,-) > K(z,+)$. It appears that if the medium were homogeneous and infinitely optically deep. the monotonic decrease would continue indefinitely toward a common limit k_∞.

(b) Under the same conditions as in (a), the values $R(z,-)$ appear to exhibit a *monotonic* increase toward a well-defined limit R_∞.

(c) The distribution functions $D(z,\pm)$ are *practically constant* with depth.

(d) The ratio of $\alpha(z)$ to $a(z)$ and hence the ratio $s(z)/\alpha(z)$, where $\alpha(z) = a(z) + s(z)$, appears to be *practically constant* with depth.

10.3 Formulation of the Shallow-Depth Model for K and R Functions

On the basis of the experimental evidence summarized in Sec. 10.2, in particular statements (c) and (d), we may adopt the two-D model of the light field as developed in Sec. 8.5. The equations forming the basis of this model have been explored in detail throughout Chapter 8. Therefore it simply remains to solve the equations of this model for the particular context at hand. We shall adopt the two-D model for decomposed light fields as given in Sec. 8.5. In addition to the assumptions leading to (67) of Sec. 8.5, we specifically emphasize that the optical medium is:

(i) Optically infinite deep.

(ii) Separable. (The ratio of $s(z)/\alpha(z)$ is invariant with depth--see experimental statement (d) of Sec. 10.2.)

(iii) Irradiated by a collimated radiance distribution of magnitude N^o incident at an angle θ_o from the normal to its upper boundary.

Formulas for $H(z,\pm)$

It follows from the two-D theory, in particular (71)-(73) of Sec. 8.5, that under the present conditions the expressions for $H(z,\pm)$ $(= H^o(z,\pm) + H^*(z,\pm))$ are:

$$H(z,-) = N^o \left[C(\mu_o,-)e^{-k_\infty z} + [\mu_o - C(\mu_o,-) \cdot e^{-\alpha z/\mu_o}\right] \tag{1}$$

$$H(z,+) = N^o \left[C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}\right] \tag{2}$$

The quantities $C(\mu_o,\pm)$, k_∞, and R_∞ and their component parts are defined in detail in Sec. 8.5. It is of interest to compare (1) and (2) with (37) and (38) of Sec. 8.6. For convenience we repeat basic formulas of Sec. 8.5; (72) of Sec. 8.5 is of the form:

$$C(\mu_o,\pm) = \frac{\sigma_\pm(\mu_o)b^*(\mp) + \sigma_\pm(\mu_o)\left[a^*(\mp) + b^*(\mp) \mp \left(\frac{\alpha}{\mu_o}\right)\right]}{\left(k_+ + \frac{\alpha}{\mu_o}\right)\left(k_- + \frac{\alpha}{\mu_o}\right)} \tag{3}$$

where $\mu_o = \cos\theta_o$ (cf. (70) of Sec. 8.5). Furthermore:

$$k_{\pm} = \frac{1}{2}\left\{\left[a^*(+) + b^*(+) - a^*(-) - b^*(-)\right] \pm \left[(a^*(+) + b^*(+) + a^*(-) + b^*(-))^2 - 4b^*(-)b^*(+)\right]\right\}^{1/2} \qquad (4)$$

and we have chosen to adopt the notation "k_∞," to point up the infinite depth of the medium. Hence:

$$k_\infty = -k_- \quad , \quad k_\infty \leq \alpha \quad , \qquad (5)$$

Finally from (102) of Sec. 8.7:

$$R_\infty = \frac{k_\infty - a^*(-)}{k_\infty + a^*(+)} \quad . \qquad (6)$$

Because of assumption (iii), the $H(z,\pm)$ depend implicitly on the quantity μ_o and to explicitly note this we would write "$H(z,\pm;\mu_o)$." If the medium is irradiated by an arbitrary radiance distribution $N^o(\mu,\phi)$ then the associated irradiances are found by an appropriate integration of the normalized forms of (1) and (2) (set $N^o = 1$ in (1), (2)):

$$H(z,\pm) = \int_{\phi=0}^{2\pi}\int_{\mu=0}^{1} H(z,\pm;\mu)N^o(\mu,\phi)\,d\mu\,d\phi \quad .$$

However, the present results can be deduced by a direct examination of (1) and (2) for an arbitrary μ_o without having to consider the general μ_o - effects as summarized in the preceding equation.

Formulas for $K(z,\pm)$

Using the definitions of the K - functions and the formulas for $H(z,\pm)$ given in (1) and (2), we have

$$K(z,-) = \frac{k_\infty C(\mu_o,-)e^{-k_\infty z} + \frac{\alpha}{\mu_o}\left(\mu_o - C(\mu_o,-)\right) e^{-\alpha z/\mu_o}}{C(\mu_o,-)e^{-k_\infty z} + \left(\mu_o - C(\mu_o,-)\right) e^{-\alpha z/\mu_o}} \qquad (7)$$

or

$$\boxed{K(z,-) = \frac{k_\infty - \frac{\alpha}{\mu_o}A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}{1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}} \quad , \qquad (8)$$

where we have written:

$$\text{"}A(\mu_o,-)\text{"} \quad \text{for} \quad \frac{C(\mu_o,-) - \mu_o}{C(\mu_o,-)} \tag{9}$$

Furthermore:

$$K(z,+) = \frac{k_\infty C(\mu_o,-)R_\infty e^{-k_\infty z} - \frac{\alpha}{\mu_o} C(\mu_o,+)e^{-\alpha z/\mu_o}}{C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}} \tag{10}$$

or

$$\boxed{K(z,+) = \frac{k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{1 - A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}} \tag{11}$$

where we have written:

$$\text{"}A(\mu_o,+)\text{"} \quad \text{for} \quad \frac{1}{R_\infty}\frac{C(\mu_o,+)}{C(\mu_o,-)} \quad . \tag{12}$$

Formulas (8) and (11) may be used to predict the depth dependence of $K(z,\pm)$. We deduce immediately from these equations the general fact that:

$$\lim_{z\to\infty} K(z,\pm) = k_\infty \quad . \tag{13}$$

Furthermore the K - functions approach this limit in a monotonic manner, as can be seen by taking their derivatives with respect to z:

$$\frac{dK(z,\pm)}{dz} = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)^2 A(\mu_o,\pm)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{\left[1 - A(\mu_o,\pm)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}\right]^2} \quad . \tag{14}$$

The monotonic behavior of $K(z,\pm)$ follows from the observation that the derivatives $dK(z,\pm)/dz$ are of *constant sign* for all z and for arbitrary choice of μ_o. In particular, the model predicts that:

$$\frac{dK(z,\pm)}{dz} > 0 \quad \text{if} \quad A(\mu_o,\pm) > 0 \left.\vphantom{\frac{a}{b}}\right\} \begin{array}{l} K \text{ increasing} \\ H \text{ concave downward} \end{array} \tag{15}$$

and:

$$\frac{dK(z,\pm)}{dz} = 0 \quad \text{if} \quad A(\mu_o,\pm) = 0 \left.\right\} \begin{array}{l} K \text{ constant} \\ H \text{ linear} \end{array} \tag{16}$$

and:

$$\frac{dK(z,\pm)}{dz} < 0 \quad \text{if} \quad A(\mu_o,\pm) < 0 \left.\right\} \begin{array}{l} K \text{ decreasing} \\ H \text{ concave upward} \end{array} \tag{17}$$

It thus appears that the model qualitatively reproduces the experimentally observed depth behavior of $K(z,\pm)$ as summarized in (a) of Sec. 10.2. The question of whether $K(z,+)$ and $K(z,-)$ increase or decrease with increasing depth is settled by evaluating the quantities $A(\mu_o,+)$ and $A(\mu_o,-)$, respectively, and applying the criteria (15)-(17). Clearly the increase or decrease of the K-functions is governed, according to the present model, by the nature of the external lighting conditions (summarized by the parameter μ_o) and the salient optical properties of the medium used in two-D models (summarized by α, k_∞, $A(\mu_o,\pm)$). Some specific illustrations of this fact are given below. For the present, we turn to the consideration of $R(z,-)$.

Formula for R(z,-)

Using the definition of $R(z,-)$ and the formulas for $H(z,\pm)$ given in (1) and (2), we have:

$$R(z,-) = \frac{C(\mu_o,-)R_\infty e^{-k_\infty z} - C(\mu_o,+)e^{-\alpha z/\mu_o}}{C(\mu_o,-)e^{-k_\infty z} + (\mu_o - C(\mu_o,-)e^{-\alpha z/\mu_o}} , \tag{18}$$

or

$$\boxed{R(z,-) = R_\infty \frac{1 - A(\mu_o,+)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}{1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z}}} \; . \tag{19}$$

Formula (19) may be used to predict the depth dependence of $R(z,-)$. We deduce immediately that, in general,

$$\lim_{z\to\infty} R(z,-) = R_\infty \quad ,$$

and that $R(z,-)$ approaches this limit in a monotonic manner, as can be seen by taking the derivatives of (19) with respect to z :

$$\frac{dR(z,-)}{dz} = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)\left(R_\infty - R(0,-)\right)\mu_o C(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}{\left[C(\mu_o,-) + (\mu_o - C(\mu_o,-))e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}\right]^2} \tag{20}$$

or

$$\frac{dR(z,-)}{dz} = \frac{R_\infty\left(\frac{\alpha}{\mu_o} - k_\infty\right)[A(\mu_o,+) - A(\mu_o,-)]e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}}{\left[1 - A(\mu_o,-)e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right)z}\right]^2} \tag{21}$$

The monotonic behavior of $R(z,-)$ follows from the observation that $dR(z,-)/dz$ is of *constant sign* for all z, and for arbitrary fixed choice of μ_o. It appears that the model can qualitatively reproduce the experimentally observed depth behavior of $R(z,-)$ as summarized in (b) of Sec. 10.2 above.

In particular, the model predicts that:

$$\frac{dR(z,-)}{dz} > 0 \quad \text{if} \quad A(\mu_o,+) > A(\mu_o,-) \Big\} \quad R \text{ increasing} \tag{22}$$

$$\frac{dR(z,-)}{dz} = 0 \quad \text{if} \quad A(\mu_o,+) = A(\mu_o,-) \Big\} \quad R \text{ constant} \tag{23}$$

$$\frac{dR(z,-)}{dz} < 0 \quad \text{if} \quad A(\mu_o,+) < A(\mu_o,-) \Big\} \quad R \text{ decreasing} \tag{24}$$

The increase or decrease of $R(z,-)$ with increasing depth is therefore governed by the relative magnitudes of $A(\mu_o,\pm)$, according to the criteria (22)-(24).

Comparisons of Experimental Data with Calculations Based on the Model

Figure 10.5 shows a graphical comparison of the experimentally determined K - function values (the crosses) with the calculated values of these functions (solid curves) using the formulas (8) and (11) deduced from the model. Table 1 gives a numerical comparison of the values. The agreement between experimental data and theory appears to be good.

Figure 10.6 shows a graphical comparison of the experimentally determined R - function values (crosses) with the calculated values of these functions (solid curve) using the formula (19) deduced from the model. Table 1 includes a numerical comparison of the values. The agreement between the computed and measured values is excellent in this case.

A word about the calculation procedure may be in order. The following values of the optical properties were used: $k_\infty = 0.178$/meter, $\alpha = 0.430$/meter. The quantities $A(\mu_o,\pm)$, for curve-fitting purposes, may be considered as constants of

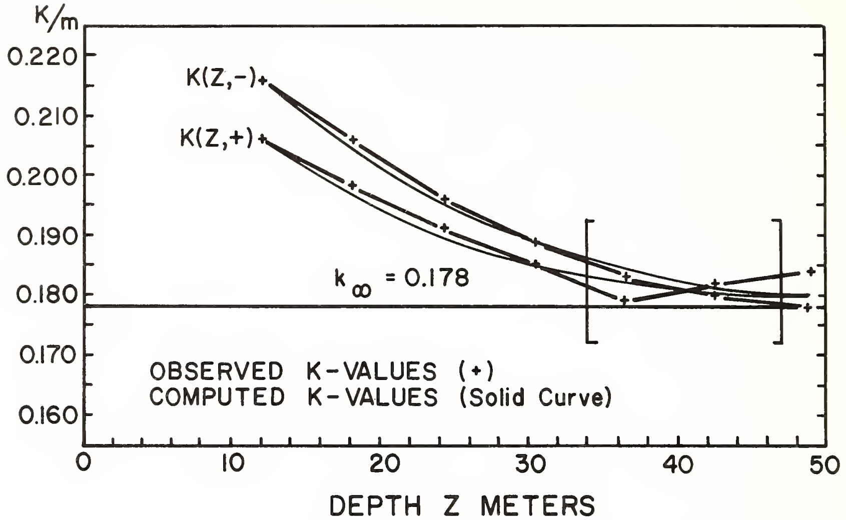


FIG. 10.5 Comparison between experimental and theoretical K-function values, as given by equations (8) and (11).

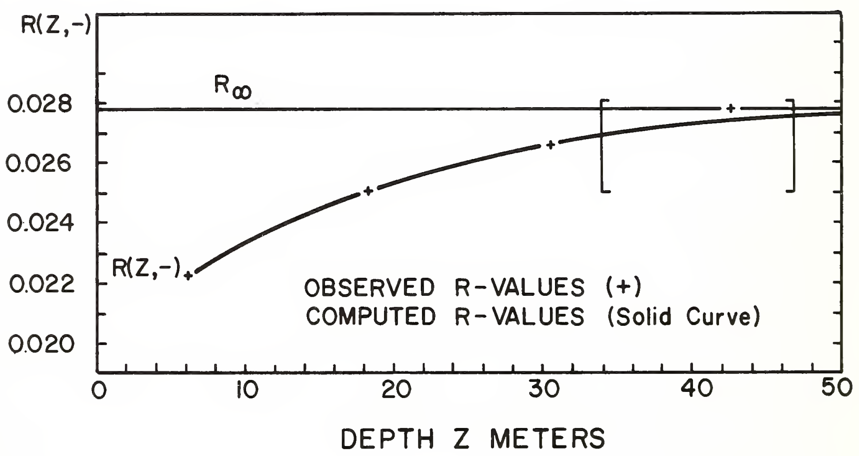


FIG. 10.6 Comparison between experimental and theoretical reflectance function values, as given by (19).

integration. Their values were therefore determined in the present by using the following boundary conditions:

$$K(12.2,-) = 0.216/\text{meter} \quad ,$$

$$K(12.2,+) = 0.206/\text{meter} \quad .$$

TABLE 1

Comparison of Calculated and Measured K and R Functions

z meters	K(z,-)		K(z,+)		R(z,-)	
	Data	Calculated	Data	Calculated	Data	Calculated
6.10	-	-	-	-	0.0221	0.0221
12.20	0.216	0.216	0.206	0.206	-	-
18.30	0.206	0.204	0.198	0.196	0.0250	0.0249
24.41	0.196	0.195	0.191	0.189	-	-
30.52	0.189	0.188	0.185	0.185	0.0266	0.0266
36.64	0.183	0.184	0.179	0.182	-	-
42.76	0.180	0.182	0.182	0.180	0.0279	0.0274
48.88	0.178	1.180	0.184	0.179	-	-
54.99	-	-	-	-	0.0258	0.0277

Note: $\mu_o = 1.582$ $A(\mu_o,+) = -1.337$ $A(\mu_o,-) = -2.141$

$R_\infty = 0.0278$ $k_\infty = 0.178/\text{meter}$ $\alpha = 0.430/\text{meter}$

$\lambda = 480 \pm 64$ mμ

A value $\mu_o = 1.583$ was found by computing the slope at $z = 12.2$ meters of the experimental $K(z,-)$ curve. This is an effective μ_o in the sense that it simulates the *noncollimated* external lighting conditions and interreflection effects at the boundary. Recall that assumption (ii) of the model (stated before (8) of Sec. 8.5) makes it strictly applicable only to media with nonreflecting boundaries irradiated by a collimated radiance distribution. In this way the lengthy integration process of the kind described above (following (6), and which is strictly necessary) was conveniently bypassed.

When the values of the constants $A(\mu_o,\pm)$ and μ_o, so found at the single depth $z = 12.2$ meters, were substituted in (8), (11), and (19), these formulas predicted a set of values of $K(z,\pm)$ and $R(z,-)$ for all other depths. These predicted values are shown in Table 1.

Hypotheses on the Fine Structure of Light Fields in Natural Hydrosols

We have seen (Sec. 10.2) that there is experimental evidence of a regular nonlinear trend in the logarithmic derivatives and the ratios of the upwelling and downwelling irradiances in near-homogeneous natural hydrosols. On the basis of this evidence, and the ability of the present mathematical model of the light field for homogeneous natural hydrosols to quantitatively reproduce these effects, we conclude that these nonlinearities are effects which may be

expected to be observed in all homogeneous natural hydrosols. We are thus led to tentatively propose the following hypotheses about the fine structure of the light field in all homogeneous source-free natural hydrosols. The part of the hypotheses concerned with the limiting behavior of the K and R functions will be proved in detail in Sec. 10.7 but is included here for completeness.

I. *The ratio of the upwelling to the downwelling irradiance, i.e., the observable reflectance function $R(z,-)$, is monotonic increasing or decreasing with increasing depth z; $R(z,-)$ always approaches a limit R_∞, which depends only on the inherent optical properties of the hydrosol and which is independent of the external lighting conditions at the upper boundary of the medium.*

II. *The logarithmic derivatives $K(z,\pm)$ i.e., the K functions for the upwelling and downwelling irradiances are monotonic increasing or decreasing with increasing depth z; $K(z,\pm)$ always approaches a common limit k_∞ which depends only on the inherent optical properties of the hydrosol and which is independent of the external lighting conditions at the upper boundary of the medium.*

On the basis of the experimental evidence cited above, and on an examination of the mathematical model of the observed phenomena, we can propose an additional hypothesis which goes on to state more specifically the depth behavior of the K- and R-functions.

III. *In all homogeneous source-free natural hydrosols:*

(a) $K(z,-) > K(z,+)$ for all $z \geq 0$.

(b) $dK(z,\pm)/dz < 0$ for all $z \geq 0$.

We immediately deduce that:

$$\frac{dR(z,-)}{dz} > 0 \quad , \tag{25}$$

which follows from (a) and general relation ((32) of Sec. 9.2)

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad . \tag{26}$$

Hypothesis III is more specific than hypotheses I and II: (b) implies hypothesis II, and Equation (25) asserts that the reflectance function $R(z,-)$ monotonically *increases* with increasing depth, thus (a) implies a specific alternate in I.

The hypothesis cited in III is actually but one of a score of possibilities. It has, however, a relatively high probability of occurring. The sense of this "probability" will be made clear below in Sec. 10.4.

10.4 Catalog of K Configurations for Shallow Depths

In order to facilitate further theoretical studies of hypothesis III in Sec. 10.3 and to anticipate alternate possibilities, we shall develop in this section a catalog of all possible K-function configurations, as predicted by the two-D model.

The catalog of K-configurations in Figs. 10.7-10.12 is a graphical listing of all ways in which $K(z,-)$ and $K(z,+)$ may approach their common limit k_∞ in various homogeneous source-free natural hydrosols. It would be of interest to try to reproduce each of the possible configuration under controlled laboratory conditions.

A *K-configuration* is defined as an ordered quadruple of the four quantities: k_∞, $K(0,+)$, $K(0,-)$, and 0. The catalog

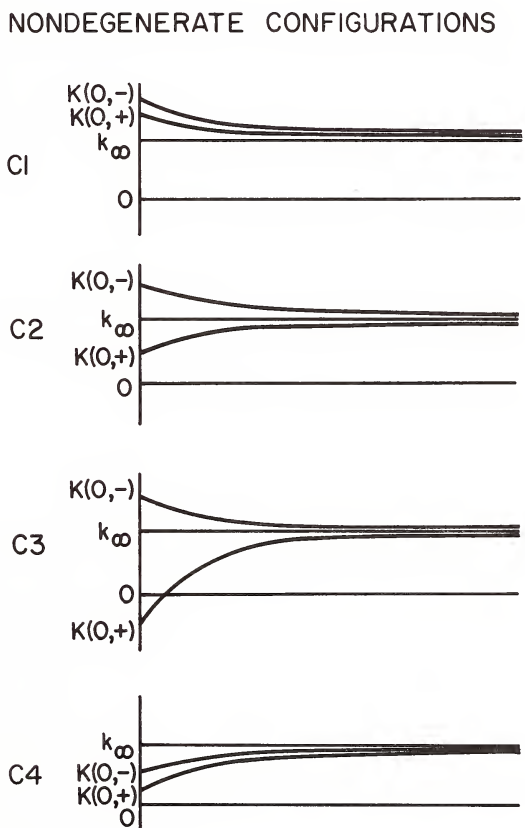


FIG. 10.7 Figures 10.7-10.12 catalog the totality of distinct depth-dependent configurations possible between the three K-functions $K(z,\pm)$, $k(z)$ in homogeneous stratified plane-parallel optical media, as deduced from the two-D theory for irradiance fields. See text for further details.

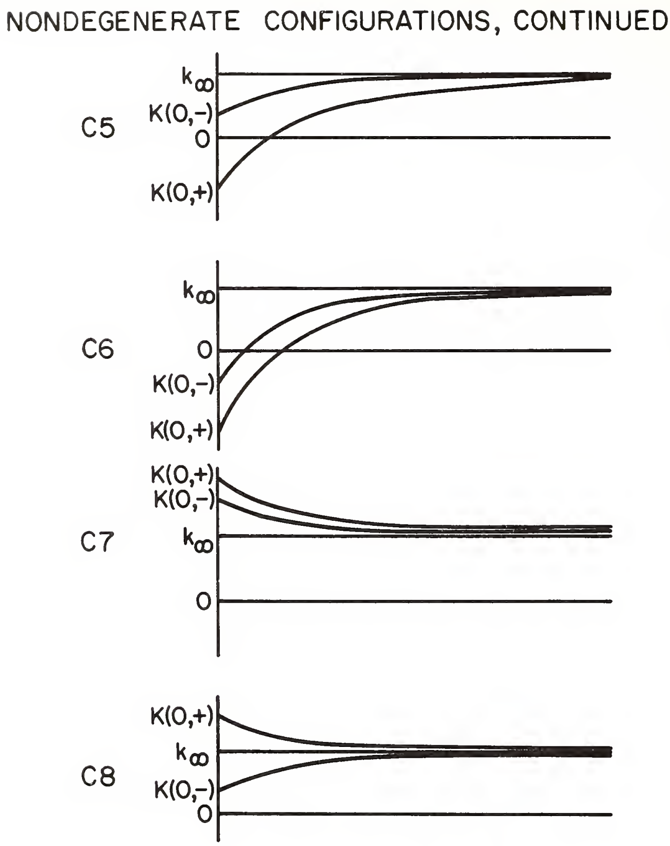


FIG. 10.8

consists of 25 configurations. These configurations are divided into three main classes:

1. Nondegenerate Configurations (9 members)
2. Degenerate Configurations
   (a) First Kind (8 members)
   (b) Second Kind (3 members)
3. Forbidden Configurations (5 members)

The *nondegenerate configuration* is defined as one in which:

$$0 \neq k_\infty \neq K(0,-) \neq K(0,+) \neq k_\infty \quad . \tag{1}$$

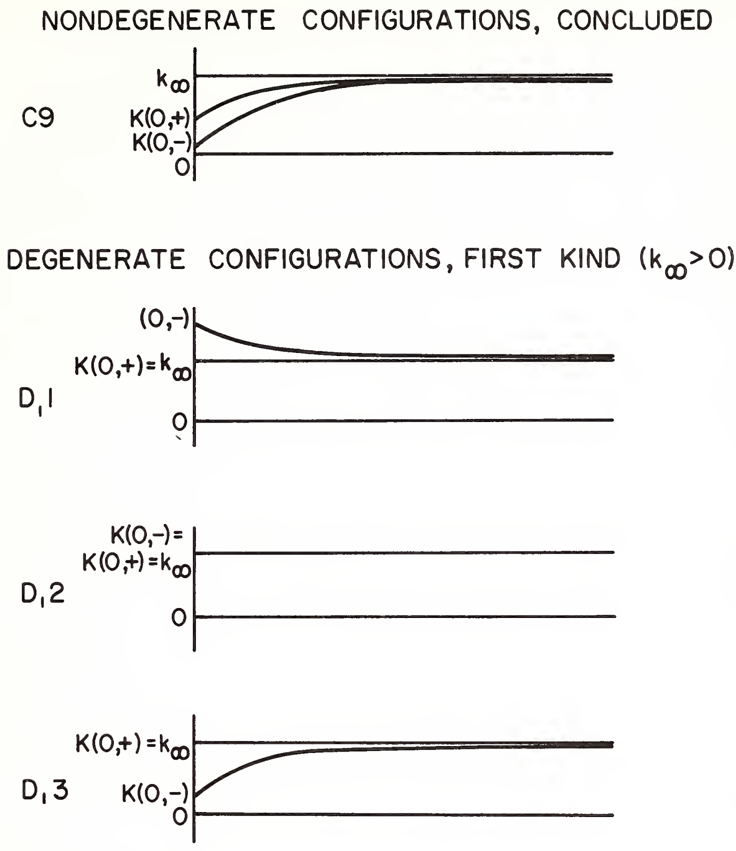


FIG. 10.9

A *degenerate configuration* is defined as one in which *at least* one of the inequalities in (1) is replaced by an equality.

A *forbidden configuration* is one in which the following basic inequality of the general theory ((30) of Sec. 9.2) is violated:

$$K(0,+)R(0,-) \leq K(0,-) \quad . \tag{2}$$

Observe that the K-configurations are defined in terms of the values of the K-functions for z = 0. This is possible because the monotonic behavior of the K and R functions, as established in 10.3, fixes their relative behavior at all depths once their initial values are known. For example, in C1 of Fig. 10.7:

$$K(0,-) > K(0,+) > k_\infty > 0 \quad .$$

DEGENERATE CONFIGURATIONS, FIRST KIND ($k_\infty > 0$) CONTINUED

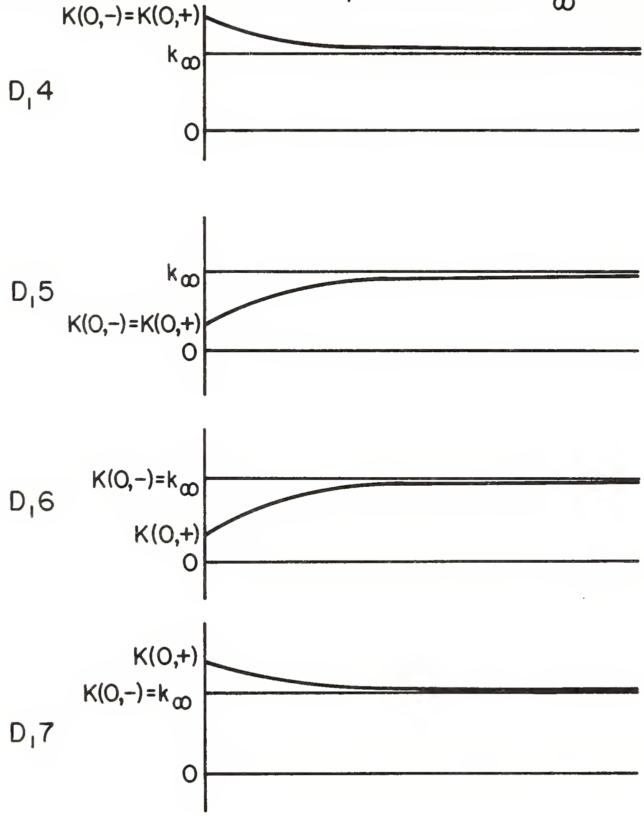
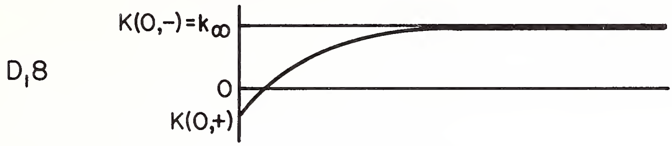


FIG. 10.10

Then since $K(z,\pm)$ must always decrease or always increase toward its limit we must have, in the present example, a *decrease* of both $K(z,+)$ and $K(z,-)$ toward k_∞. Furthermore, since $R(z,-)$ also exhibits a fixed monotonic behavior for all z, C1 must have--by virtue of (26) of Sec. 10.3-- $K(z,-) > K(z,+)$ for all z. Similar arguments show that all the configurations are well-defined in terms of the initial values of the K-functions. Knowing the initial magnitudes of $K(0,+)$ and $K(0,-)$ therefore fixes each configuration for all z. The general relation (26) of Sec. 10.3 may be used to determine whether $R(z,-)$ increases or decreases for a particular K-configuration.

We now give evidence that a configuration in which $K(z,-) > K(z,+)$ is preferred to one with $K(z,-) < K(z,+)$. We begin by noting that the most unlikely configuration is $D_2 3$ which is associated with nonabsorbing (purely scattering)

DEGENERATE CONFIGURATIONS, FIRST KIND ($k_\infty > 0$) CONCLUDED



DEGENERATE CONFIGURATION, SECOND KIND ($k_\infty = 0$)

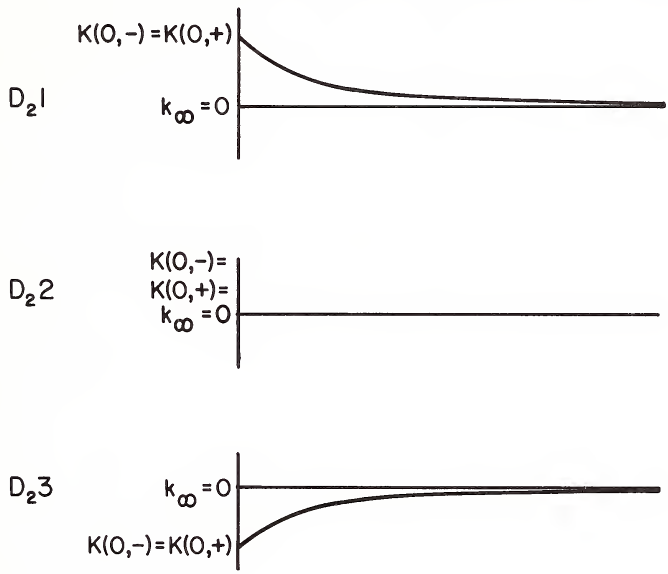


FIG. 10.11

media with $\sigma_+(\mu_0) \neq 0$ for all μ_0. In such media, we have by (18) of Sec. 8.8,

$$\frac{d\overline{H}(z,+)}{dz} = a(z)h(z) = 0 \quad .$$

Hence, for some constant C,

$$\overline{H}(z,+) = H(z,+) - H(z,-) = C$$

for all $z \geq 0$. The preceding formula implies that

$$K(z,+) = K(z,-)$$

for all $z \geq 0$, whence by (26)

$$R(z,-) = R_\infty \leq 1$$

for all $z \geq 0$. Now from the fixed value of $\overline{H}(z,+)$ and (1)

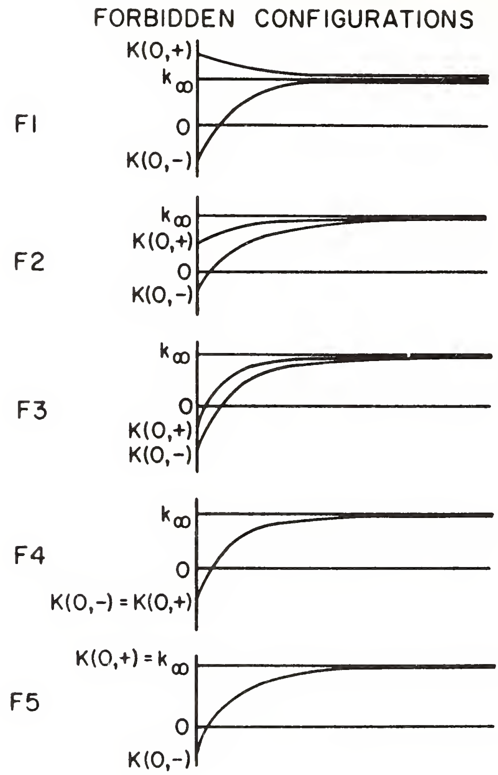


FIG. 10.12

and (2) of Sec. 10.3, it must follow that

$$R_\infty = 1 \quad , \quad k_\infty \geq 0 \quad \text{and} \quad C(\mu_o,-) = C(\mu_o,-) - \mu_o$$

Hence we have the following representations of $H(z,\pm)$:

$$H(z,-) = N^o \left[C(\mu_o,-)e^{-k_\infty z} - C(\mu_o,+)e^{-sz/\mu_o} \right]$$

$$H(z,+) = N^o \left[C(\mu_o,-)e^{-k_\infty z} - C(\mu_o,+)e^{-sz/\mu_o} \right]$$

that is,

$$H(z,+) = H(z,-)$$

for every $z \geq 0$. Moreover, a study of (14)-(16) of Sec. 8.5 shows that if $k_+ = k_-$ (as must be the case in view of $K(z,+) = K(z,-)$ deduced above), then necessarily $k_\infty = 0$. (Here we are identifying $K(z,\pm)$ with $\pm\, k_\pm$ respectively. The basis for this is (31) of Sec. 9.2.) It follows that, under the

present circumstances we must have:

$$H(z,\pm) = N^{o}\left[C(\mu_{o},-) - C(\mu_{o},+)e^{-sz/\mu_{o}}\right]$$

Now, in forward scattering media, an examination of (3) of Sec. 10.3 shows that it is more likely to have

$$\mu_{o} \geq C(\mu_{o},-)$$

than

$$\mu_{o} < C(\mu_{o},-)$$

Hence if a medium is such that $a = 0$, and that it exhibits backward scattering, then $H(z,\pm)$, by the preceding representations for $H(z,\pm)$, *increase* with increasing z, so that $K(0,+) = K(0,-) < 0$. This unlikely state of affairs is represented by $D_2 3$ in Fig. 10.11; hence the configuration $D_2 3$ is very low on the list of likely configurations encountered in nature, as was to be shown.

Next, if the volume absorption coefficient is very slightly increased from 0 to $a > 0$, then the resulting effect is such that

$$k_{\infty} > 0$$

$$R_{\infty} < 1 \quad .$$

Furthermore, by (13) of Sec. 8.5, we would expect that $K(z,-) \neq K(z,+)$. Hence we would expect that

$$k_{\infty} > 0 > K(0,-) > K(0,+)$$

which is represented by configuration C6 in Fig. 10.8 (see configuration F3 which shows that the reverse inequality between $K(0,+)$ and $K(0,-)$ is impossible). As the value of the volume absorption coefficient a is allowed to increase a bit more, $K(0,-)$ and $K(0,+)$ move upward on the vertical axis, maintaining the above inequality as they approach and assume configuration C5. At this point, as α is further increased, the configuration assumed depends on the external lighting conditions and the volume scattering function σ. If σ is highly anisotropic with high forward scattering values and small backward scattering values, as is the case in most natural hydrosols, then the configurations C4, C3, C2, and C1 are most likely to be realized on the basis of various simple models obtained by assuming the appropriate forms for σ. Thus the phrase "configuration X is more probable than configuration Y," means that the values of the optical parameters associated with configuration X are more likely to be observed in nature than those associated with Y. This ordering of the likelihood of occurrence of optical parameters is suggested by experimental evidence. The configurations therefore are roughly in the order of decreasing probability of occurrence.

The discussion will not go into further detail on the catalog of these configurations. We merely mention that various special models can be obtained by assuming specific, but simple, forms of σ. These easily yield most of the 20 possible configurations. These forms for σ are:

(i) $\sigma(\zeta;\zeta') = \sigma_+\delta(\zeta-\zeta') + \sigma_-\delta(\zeta+\zeta')$, where σ_+,σ_- are fixed constants, and δ is the Dirac delta function (Stick model).

(ii) $\sigma(\zeta;\zeta') = s/4\pi$ where s is the volume total scattering coefficient (Ball model).

Some Special Fine Structure Relations

The models developed in Sec. 10.3 may be used to answer questions of the following kind.

1. What quantitative estimates can be made of the differences:

$$K(0,-) - K(0,+)$$
$$K(0,-) - k_\infty$$
$$K(0,+) - k_\infty \quad ?$$

2. What quantitative estimates can be made of the differences

$$R(0,-) - R_\infty \quad \text{knowing either } R_\infty \text{ or } R(0,-)?$$

3. What can be said about the relative magnitudes of

$$K(z,+),\ K(z,-),\ \text{and } k(z)?$$

4. If $K(0,\pm) < 0$, this implies that in real media $H(z,\pm)$ (respectively) has a maximum at some depth z_{max}. What estimates can be made of z_{max}?

The numerical examples giving the general answers to the first three questions show that the variations of $K(z,\pm)$ and $R(z,-)$ in the hydrosol of Sec. 10.3 are not less than the expected errors of the observed data. Therefore, detailed precise measurements of the light field in natural hydrosols should be generally expected to exhibit the nonlinearity in the H, K, and R plots; the presence of these nonlinearities and their classification (by means of the appropriate catalog of the observed K-configuration) could form part of the detailed description of the hydrosols.

Answer to Question 1. From (8) and (11) of Sec. 10.3, after simplification, we have

$$K(0,-) - K(0,+) = \frac{\left(\frac{\alpha}{\mu_o} - k_\infty\right)\left[A(\mu_o,+) - A(\mu_o,-)\right]}{\left[1 - A(\mu_o,-)\right]\left[1 - A(\mu_o,+)\right]} \qquad (3)$$

As an example of the use of this formula we use the values of $A(\mu_o,\pm)$ obtained in the computations for Table 1 in Sec. 10.3 above, whence:

$$K(0,-) - K(0,+) = 0.010/\text{meter}.$$

In addition to (3), we have:

$$K(0,\pm) - k_\infty = - \frac{\left(\frac{\alpha}{\mu_0} - k_\infty\right) A(\mu_0,\pm)}{1 - A(\mu_0,\pm)} \tag{4}$$

For the case of the present medium, we estimate that:

$$K(0,-) - k_\infty = 0.064/\text{meter}$$

$$K(0,+) - k_\infty = 0.054/\text{meter}.$$

Answer to Question 2. From (19) of Sec. 10.3 we have:

$$R(0,-) - R_\infty = \frac{R_\infty \left[A(\mu_0,-) - A(\mu_0,+)\right]}{1 - A(\mu_0,-)}$$

In the present case, we know R_∞ and $A(\mu_0,\pm)$. This leads to the estimate:

$$R(0,-) - R_\infty = - 0.007$$

for the present medium. Therefore the spread $R(0,-) - R_\infty$ of $R(z,-)$ values in the present hydrosol is on the order of 30 percent of $R(0,-)$.

Answer to Question 3. The definition of $k(z)$ is exactly analogous to the definition of $K(z,\pm)$:

$$k(z) = - \frac{1}{h(z)} \frac{dh(z)}{dz} \quad .$$

To obtain an expression involving $K(z,\pm)$ and $k(z)$, we use the notion of the distribution function which links $H(z,\pm)$ and the corresponding components $h(z,\pm)$ of $h(z)$:

$$h(z) = h(z,-) + h(z,+) \quad ,$$

where $h(z,\pm)$ (as in (11) of Sec. 2.7) is the scalar irradiance associated with the downwelling (-) or upwelling (+) stream of radiant energy. From the definitions of $D(z,\pm)$:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} \quad ,$$

and that of $K(z)$, we have:

$$
\begin{aligned}
k(z) &= -\frac{1}{h(z)}\frac{d}{dz}\Big[D(z,-)H(z,-) + D(z,+)H(z,+)\Big] \\
&= \frac{1}{h(z)}\Big[D(z,-)H(z,-)K(z,-) + D(z,+)H(z,+)K(z,+)\Big] \\
&\quad - \frac{1}{h(z)}\left[\frac{dD(z,-)}{dz}H(z,-) + \frac{dD(z,+)}{dz}H(z,+)\right] \quad .
\end{aligned}
$$

This is the exact representation of $k(z)$ in terms of the D and K functions.

According to the present model, however,

$$\frac{dD(z,\pm)}{dz} = 0 \quad .$$

This assumption is in good agreement with experimental fact (re: Fig. 10.2). For the present medium,

$$\frac{dD(z,\pm)}{dz} \sim 0.001/\text{meter},$$

as may be verified from Fig. 10.2. The number 0.001 is an upper limit of the derivative values over the indicated depth range. Since $K(z,\pm)$ are usually determined to about 10^{-3} per meter, the contribution to $k(z)$ by the terms containing the derivatives of $D(z,\pm)$ is not significant. Hence we may write:

$$k(z) = \gamma(z)K(z,-) + [1 - \gamma(z)]K(z,+) \quad ,$$

where

$$\gamma(z) = \frac{h(z,-)}{h(z)} < 1 \quad .$$

This representation of $k(z)$ shows that $k(z)$ is expected to be between $K(z,-)$ and $K(z,+)$, regardless of the algebraic signs of $K(z,+)$ and $K(z,-)$. As an example, let $z = 30$ meters. Computations from the data yield the value $k(30) =$ $= 0.187$/meter. Therefore we have, as expected,

$$0.188/\text{meter} - K(30,-) > 0.187 = k(30) > K(30,+) = 0.185 \quad .$$

Answer to Question 4. Configurations C3, C5, C6, $D_1 8$, and $D_2 3$ exhibit all possible ways in which either $K(0,+)$ or $K(0,-)$ may be negative. All except the last configuration exhibits a finite depth z_{max} at which $K(z_{max},+) = 0$ or $K(z_{max},-) = 0$. In these cases, z_{max} is the abscissa of the maximum point on the corresponding H-curve (observe that z_{max} may differ for $K(z,+)$ and $K(z,-)$).

An estimate of z_{max} for the upwelling (+) or downwelling (-) stream can be made directly from (8) and (11) of Sec. 10.3 by setting:

$$K(z_{max},\pm) = 0 \quad ,$$

and solving for z_{max}. Thus, from (8) and (11):

$$0 = K(z_{max},\pm) = \frac{k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}}}{1 - A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}}} ,$$

from which:

$$0 = k_\infty - \frac{\alpha}{\mu_o} A(\mu_o,\pm) e^{-\left(\frac{\alpha}{\mu_o} - k_\infty\right) z_{max}} .$$

Solving for z_{max}:

$$z_{max}(\pm) = \frac{\ln\left[\frac{\alpha}{k_\infty \mu} A(\mu_o,\pm)\right]}{\frac{\alpha}{\mu_o} - k_\infty} ,$$

where the plus sign refers as usual to the upwelling stream and the minus sign refers to the downwelling stream.

The criterion for the existence of a positive z_{max} value is evidently:

$$\frac{\alpha}{k_\infty \mu_o} A(\mu_o,\pm) > 1 .$$

If, in particular, the argument of the natural logarithm is positive but less than unity then z_{max} has a negative y , sign, which means that $H(z,+)$ (or $H(z,-)$) has no maximum value. In this case the irradiance simply decreases monotonically for all depths $z > 0$. The conditions under which the preceding inequality holds have yet to be fully explored.

Conclusion

The discussions of the preceding four sections have shown that there exist in natural optical media certain well-ordered, calculable tendencies of growth and decay of the irradiance fields, and their K-functions, with respect to depth. Furthermore, only a few of a large set of theoretical possibilities (configurations C1, C2, C3, and C4) are most likely to be observed in nature, and then principally in shallow depth regions (less than 20 attenuation lengths) of homogeneous hydrosols with calm surfaces.

10.5 A General Proof of the Asymptotic Radiance Hypothesis

In this and the following section we present two proofs of the asymptotic radiance hypothesis, that is, the conjecture concerning the shape of the radiance distribution and its behavior at great depths in natural hydrosols. The two proofs differ in their starting assumptions. The proof given in this section is completely general and uses only the mildest assumptions concerning the functions used to describe natural light fields and their supporting media. The proof is based on the natural mode of solution of the equation of transfer (Chapter 5) and thereby has, despite its analytic complexity and length, the virtue of using only intuitively simple constructs in its development and which, furthermore, begins and ends with directly observable concepts, namely the radiance K-function, the volume attenuation function α, and volume scattering function σ. The proof given in Sec. 10.6 is considerably simpler than that offered in this section because it is assumed at the outset that scalar irradiance $h(z)$ in natural media eventually decreases exponentially with depth, a fact which is *not* assumed and is eventually derived in the present longer proof. This is quite a reasonable assumption, however, being born and sustained on inspection of much experimental evidence, and most students of the subject will thus be content with the shorter arguments of Sec. 10.6. However, those who wish to see the argument developed from first principles, are invited to read on below. The reader wishing only an overview of the arguments of the present proof need only read on through the discussion of (18) below.

At the conclusion of the chapter we will have accumulated four essentially distinct types of proof of the asymptotic radiance hypothesis, namely that given via the characteristic representation of N in Sec. 7.10, those of the present and subsequent section, and finally that in the closing observations of Sec. 10.7. Further, perhaps more elegant proofs should be forthcoming from the functional equations developed throughout Chapter 7. These are left for interested students to pursue.

Introduction

The asymptotic radiance hypothesis was formulated in the field of experimental radiative transfer dealing with the penetration of natural light into the oceans and deep lakes. It may be stated as follows: The form of the radiance distribution about a point in an optical medium approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions at the upper boundary of the medium and which depends only on the inherent optical properties of the medium. Some relatively early references to the hypothesis may be found in the experimental papers of Whitney [315], [316], Poole [209], and Lenoble [154]. Some recent theoretical discussions for particular cases may be found Herman and Lenoble [107], [108]. Subsequently, the mathematical problem underlying the hypothesis took on meaning in a wider set of contexts such as astrophysical optics and neutron transport theory. However, the statement of the hypothesis for these contexts is essentially the same.

In this section a proof of the hypothesis is given for a rather wide class of inhomogeneous spaces known as *eventually separable* spaces, a term which is defined in detail below. The discussion is designed so that the main results are also applicable to the astrophysical and neutron contexts. The approach used is direct in the sense that it is based on a study of the natural mode of solution of the equation of transfer rather than first solving the equation for particular cases and then inspecting the properties of the resulting solutions. Furthermore, the quantities introduced in the study are for the most part directly observable quantities, a feature which reflects the experimental origins of the problem and which keeps sight of possible practical applications of the asymptotic radiance hypothesis. In this way the discussion complements a different approach to the problem, namely the formal-solution approach initiated by Chandrasekhar [43] and extended by Kuscer [147] in the radiative transfer context, and also considered, for example by Davison [62] in the neutron transport context. In particular, the present discussion shows in terms of directly observable quantities that when an asymptotic radiance distribution exists in a medium, it is represented by a formal-solution distribution and is approached in a continuous way by the natural distributions as depth is increased in the medium. Two illustrations of this fact are given. One is based on tables compiled from theoretical calculations made in the neutron transport context, the other is drawn from an experiment which documented the light field in a natural hydrosol. These illustrations will be considered later.

The practical consequences of the asymptotic radiance theorem are many. They take on especial utility in the field of geophysical optics. While an exhaustive discussion of these consequencies is out of place here (see Secs. 10.7 and 10.8), we should observe that the classical two-flow equations of the light field (Chapter 8), are accurate and become exact with increasing depth whenever the hypothesis holds. This results in an enormous simplification of the standard experimental procedures dealing with the determination of the optical properties of natural hydrosols. Finally, the present method allows a means of estimating, with respect to a given preassigned criterion, the optical depth at which the asymptotic distribution has been attained (Sec. 10.7).

Preliminary Definitions

We begin with the general source-free equation of transfer for the radiance function N on a general isotropic* space X as used in geophysical optics ((14) of Sec. 3.15),

---

*The arguments developed below go through with minor changes also for nonisotropic media. However, such additional generality does not add materially to the theoretical or practical consequences of the asymptotic radiance hypothesis, and is therefore not postulated at this time.

$$\frac{dN(x,\xi)}{dr} = \xi \cdot \nabla N(x,\xi) = -\alpha(x)N(x,\xi) + N_*(x,\xi) \qquad (1)$$

and recast it into a form which will be most suitable for the present discussion, and which will insure the widest domain of applicability of the present results to related fields such as astrophysical optics and neutron transport theory. Here:

$$N_*(x,\xi) = \int_\Xi N(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$= \int_\Xi \sigma(x;\xi;\xi')N(x,\xi')\,d\Omega(\xi')$$

represents the path function N_*; and is written twice, as shown, so as to point up the isotropy of X (re Definition 3 of Sec. 7.12). The volume scattering function is σ, the unit sphere of direction vectors ξ is Ξ, and the attenuation function is α.

The present problem is meaningful only in the steady state case, and is most immediately concerned with emission-free arbitrarily stratified plane-parallel media with constant index of refraction. These conditions have been adopted in (1). The introduction of the plane-parallel geometry into (1) results in the usual equation:

$$-\xi \cdot \mathbf{k}\,\frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad , \qquad (2)$$

where $\mathbf{k}$ is the unit outward normal to the plane-parallel medium $X(0,\infty)$. For the present mathematical discussion we observe that the radiance function N is defined on the domain $Z \times \Xi$, where Z is the set of nonnegative real numbers. Furthermore, $\theta = \text{arc}\cos(\xi \cdot \mathbf{k})$, so that $\xi \in \Xi$ may be represented as usual by a pair of angles $(\theta,\phi) \in \Xi$ (re: Sec. 2.4). Finally, we adopt the parameters μ and $\tau(z)$, where we have written:

$$\text{"}\mu\text{"} \quad \text{for} \quad \xi \cdot \mathbf{k} \quad ,$$

and:

$$\text{"}\tau(z)\text{"} \quad \text{for} \quad \int_0^z \alpha(z')\,dz' \quad ,$$

Recall (Sec. 7.12) that the phase function p, as used in astrophysics, is related to σ by:

$$p = 4\pi\sigma/\alpha \quad .$$

With these notations, (2) takes the form:

$$\boxed{\mu \frac{dN(\tau,\mu,\phi)}{d\tau} = N(\tau,\mu,\phi) - N_q(\tau,\mu,\phi)} \quad , \tag{3}$$

where we have written:

"$N_q(\tau,\mu,\phi)$" for $\frac{1}{4\pi}\int_\Xi p(\tau;\mu,\phi;\mu',\phi')N(\tau,\mu',\phi')d\mu'd\phi'$

which defines the equilibrium radiance function N_q. Thus (1) reduces, under the above assumptions--which will be considered in force in the sequel--to the standard form for the equation of transfer in plane-parallel media.

The discussion will require consideration of the following scattering-order decomposition of (3):

$$\mu \frac{dN^j(\tau,\mu,\phi)}{d\tau} = N^j(\tau,\mu,\phi) - N_q^j(\tau,\mu,\phi) \quad , \quad j = 1, 2, \ldots, \tag{4}$$

where we have written:

"$N_q^j(\tau,\mu,\phi)$" for $\frac{1}{4\pi}\int_\Xi p(\tau;\mu,\phi;\mu',\phi')N^{j-1}(\tau,\mu',\phi')d\mu'd\phi'$,

(5)

and where N^j and N_q^j are positive valued radiance functions on $Z \times \Xi$ which refer to radiant flux which has been scattered precisely j times, so that (as in (1) of Sec. 5.4) we have the definitional identity:

$$N(\tau,\mu,\phi) = \sum_{j=o}^{\infty} N^j(\tau,\mu,\phi) \quad . \tag{6}$$

As outlined in the introduction, the present discussion employs, whenever possible, directly observable quantitites. From the point of view of the experimenter, the depth dependence of the radiance distribution $N(\tau,\cdot,\cdot)$ on Ξ, $\tau \geq 0$, is most conveniently studied by means of the associated functions $K(\tau,\cdot,\cdot)$ on Ξ defined by writing (re: (35) of Sec. 9.2):

$$\text{"}K(\tau,\mu,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N(\tau,\mu,\phi)} \frac{dN(\tau,\mu,\phi)}{d\tau} \quad . \tag{7}$$

The present discussion will also require consideration of the function $K_q(\tau,\cdot,\cdot)$ on Ξ, $\theta \geq 0$, defined by writing

$$\text{"}K_q(\tau,\mu,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N_q(\tau,\mu,\phi)} \frac{dN_q(\tau,\mu,\phi)}{d\tau} \quad . \tag{8}$$

Similarly, we write:

$$"K^j(\tau,\mu,\phi)" \quad \text{for} \quad -\frac{1}{N^j(\tau,\mu,\phi)}\frac{dN^j(\tau,\mu,\phi)}{d\tau}, \; j = 0,1,\ldots, \; \tau > 0, \tag{9}$$

and:

$$"K_q^j(\tau,\mu,\phi)" \text{ for } -\frac{1}{N_q^j(\tau,\mu,\phi)}\frac{dN_q^j(\tau,\mu,\phi)}{d\tau}, \quad j = 1,2,\ldots, \; \tau \geq 0 . \tag{10}$$

Finally, corresponding to:

$$"N_q^{(n)}(\tau,\mu,\phi)" \quad \text{for} \quad \sum_{j=1}^{n} N_q^j(\tau,\mu,\phi) \quad , \tag{11}$$

we write:

$$"K_q^{(n)}(\tau,\mu,\phi)" \text{ for } -\frac{1}{N_q^{(n)}(\tau,\mu,\phi)}\frac{dN_q^{(n)}(\tau,\mu,\phi)}{d\tau}$$

$$= \left(\frac{\sum_{j=1}^{n} N_q^j(\tau,\mu,\phi)K_q^j(\tau,\mu,\phi)}{\sum_{j=1}^{n} N_q^j(\tau,\mu,\phi)}\right) \tag{12}$$

where, of course:

$$\lim_{n\to\infty} N_q^{(n)}(\tau,\mu,\phi) = N_q(\tau,\mu,\phi) \quad . \tag{13}$$

The equations which govern the behavior of the K-functions play a central role in what follows. The equations have the outward appearance of Riccati differential equations, and it will actually be possible to use to advantage some of the well-known properties of such differential equations.

It is easy to verify the following formulas with the help of (3) and the definitions of K and K_q (see, e.g., (19) of Sec. 10.7 or (7) of Sec. 11.2):

$$\boxed{\frac{dK(\tau,\mu,\phi)}{d\tau} = \left[K(\tau,\mu,\phi) - K_q(\tau,\mu,\phi)\right]\left[K(\tau,\mu,\phi) + \frac{1}{\mu}\right]} . \tag{14}$$

Furthermore, from (4) and the definition of K^j and K^j_q :

$$\boxed{\frac{dK^j(\tau,\mu,\phi)}{d\tau} = \left[K^j(\tau,\mu,\phi) - K^j_q(\tau,\mu,\phi)\right]\left[K^j(\tau,\mu,\phi) + \frac{1}{\mu}\right] \quad j = 1,\ 2,\ \ldots,\quad \tau \geq 0} \tag{15}$$

We now can give the motivation for the preceding adoptions of the K - functions in the present approach to the asymptotic radiance problem. Suppose there is some depth in the medium below which the functions $K(\tau,\cdot,\cdot)$, $\tau \geq \tau_0$ are constant functions on Ξ, and whose values are equal to a fixed number k_∞. Then we may write, for all $\tau \geq \tau_0$:

$$N(\tau,\mu,\phi) = N(0,\mu,\phi)\ \exp\left\{-\int_0^{\tau_o} K(\tau',\mu,\phi)\,d\tau' - \int_{\tau_o}^{\tau} K(\tau',\mu,\phi)\,d\tau'\right\}$$

$$= N(\tau_o,\mu,\phi)\ \exp\left\{-(\tau - \tau_o)k_\infty\right\} \quad .$$

Thus if we write:

$$\text{"}g(\mu,\phi)\text{"} \quad \text{for} \quad N(\tau_o,\mu,\phi)\exp\left\{\tau_o k_\infty\right\}$$

we have the following equivalent way of representing the distributions:

$$\boxed{N(\tau,\mu,\phi) = g(\mu,\phi)\ \exp\left\{-\tau k_\infty\right\}} \quad . \tag{16}$$

Relation (16) is the starting point of the classical formal procedures referred to above which lead to the determination of a specific radiance distribution g on Ξ. From the point of view of the present approach, however, (16) is an incidental end rather than a means. That is, we will be concerned with the determination of a class of spaces in which the radiance distributions tend continuously to a structure of the kind summarized in (16), and thus as a matter of course, determine a class of spaces in which such a formal procedure for the asymptotic radiance distribution is meaningful.

The preceding heuristic argument leading to (16) supplies the motivation for the following definition of an asymptotic radiance distribution: An *asymptotic radiance distribution* is said to exist if (i) $\lim_{\tau\to\infty} K(\tau,\mu,\phi)$ (henceforth denoted by "$K_\infty(\mu,\phi)$") exists for each $(\mu,\phi) \in \Xi$ and (ii) $K_\infty(\cdot,\cdot)$ is a constant function on Ξ .

It is quite possible for condition (i) of the preceding definition to hold, while condition (ii) does not hold. This state of affairs is encountered, for example, in space in which $s(\tau)/\alpha(\tau) = 0$ for all $\tau \geq 0$, where

$$s(\tau) = \int_{\Xi} \sigma(\tau;\mu',\phi';\mu,\phi)d\mu' d\phi'$$

i.e., in purely absorbing media. However, such spaces are clearly trivial from the present point of view.

Formulation of the Problem

In order to keep the usual operations on the phase function p meaningful, we will assume, as a matter of course, that p is piecewise continuously differentiable with respect to τ, and that p is continuous on $\Xi \times \Xi$ for each $\tau \geq 0$. Furthermore, we will require that the boundary radiance function $N^{o}(0,\cdot,\cdot)$ on Ξ_- be a nonnegative valued, nontrivial integrable function with respect to the measure Ω. Here, we have

$$\Xi_- = \left\{ \xi : \xi \cdot \mathbf{k} \leq 0 \right\} = \left\{ (\mu,\phi) : -1 \leq \mu \leq 0 \right\} .$$

This Ξ_- differs slightly from that usually used by now including all ξ such that $\xi \cdot \mathbf{k} = 0$. In addition, we define Ξ_+ as the complement of Ξ_- with respect to Ξ. Finally, we observe that subsets of Ξ of solid angle measure zero are of no physical interest, and have no effect on calculations which employ integrations over such sets. For example, the set of all ξ such that $\xi \cdot \mathbf{k} = 0$ is of zero Ω measure. This motivates the following standing assumption to hold throughout this section. Whenever a function f on Ξ is constant on Ξ except for a subset Ξ_0 of Ω measure zero, we shall assume that f is replaceable by a constant function $\hat{f}$ on Ξ such that $f(\xi) = \hat{f}(\xi)$ for every ξ in $\Xi - \Xi_0$. We call $\hat{f}$ the (constant) *extension* of f from $\Xi - \Xi_0$ to Ξ.

A *separable medium* is one in which the phase function is independent of position (re: Sec. 7.12). The term "separable" is used to suggest the multiplicative uncoupling of position and directional dependence that σ undergoes in such spaces: $\sigma(x;\xi;\xi') = \alpha(x)p(\xi;\xi')/4\pi$. Separable media form a class of harmlessly inhomogeneous spaces. From the point of view of the equation of transfer (3), such spaces are homogeneous. The present discussion can be carried out in a rather wide class of nonseparable spaces which we will call "eventually separable." We will say that a semi-infinite stratified plane-parallel medium $X(0,\infty)$ is *eventually separable* if, (i) the phase function p on $Z \times \Xi \times \Xi$ has the form $p(\tau;\mu,\phi;\mu',\phi') = p_\infty(\mu,\phi;\mu',\phi') + \phi(\tau;\mu,\phi;\mu',\phi')$ such that p_∞ is independent of τ and not identically zero on $\Xi \times \Xi$, and (ii) $\phi \to 0$ (the zero function) uniformly on $\Xi \times \Xi$, as $\tau \to \infty$.

We can now state the main result: *an asymptotic radiance distribution exists in every plane-parallel medium if and only if the medium is eventually separable.* This statement is understood to hold in media whose equation of transfer is given by (3), and whose boundary conditions are given as above. All of the effort below will be devoted to proving the sufficiency of the eventually separable condition. Simple counter examples show that if a space is not eventually separable, the asymptotic radiance distribution necessarily does not exist. (For example, stack in alternate layers purely absorbing and purely scattering plane parallel media.)

We close this preliminary discussion by making some observations on the K - functions which will be required below. First we observe that from (3), if $\mu = 0$, then $N(\tau,0,\phi) = N_q(\tau,0,\phi)$. Hence, for all $\tau > 0$, $K(\tau,0,\phi) = K_q(\tau,0,\phi)$. Secondly, for each $j,\tau > 0$, $N^j(\tau,\cdot,\cdot)$ is bounded away from zero, is continuous on the compact set Ξ, and hence is uniformly continuous on Ξ and Ξ_-. A similar observation holds for $N_q^j(\tau,\cdot,\cdot)$, $j = 1, 2, \ldots$. It follows that $K_q(\tau,\cdot,\cdot)$ and $K_q^j(\tau,\cdot,\cdot)$ are uniformly continuous on Ξ and Ξ_- for all $\tau > 0$ and $j = 1, 2, \ldots$. Finally, from (3) and (4) and the definitions of K and K^j,

$$K(\tau,\mu,\phi) + \frac{1}{\mu} < 0 \quad , \tag{17}$$

$$K^j(\tau,\mu,\phi) + \frac{1}{\mu} < 0 , \quad j = 0, 1, \ldots , \tag{18}$$

for all $(\mu,\phi) \in \Xi_-$ and all $\tau > 0$. Properties (17) and (18) are particularly useful in conjunction with (14) and (15). For example if $K(\tau,\mu,\phi) \lessgtr K_q(\tau,\mu,\phi)$, then by (14) and (17) it follows that $dK(\tau,\mu,\phi)/d\tau \gtrless 0$, showing that in general K always *tends* toward the equilibrium function K_q. This is a useful fact in practice. Whether or not $K \to K_q$ as $\tau \to \infty$ depends on the relative sizes of $K_q(\mu,\phi) = \lim_{\tau\to\infty} K(\tau,\xi,\phi)$. and $-(1/\mu)$. It follows directly from the properties of the Riccati equation (cf. e.g., [116, p. 312]), that $K \to \min\{K_q(\mu\,\phi), -(1/\mu)\}$ assuming of course that $K_q(\mu,\phi)$ exists. A similar set of remarks holds for (15) and (18).

The Functions P,Q,R

In order to insure that the main sequence of arguments is uninterrupted by the development of certain required auxiliary relations, these auxiliary relations are gathered here for ready reference.

The first relation needed below gives the connection between the downwelling j - scattered flux at level $\tau > 0$ and the upwelling scattered flux at level τ:

$$N^{j+1}(\tau,\mu,\phi) = \frac{1}{\mu} \int_{\Xi_-} P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi' \tag{19}$$

for all $(\mu,\phi)\varepsilon\ \Xi_+$, and where we have written

$$\text{"}P(\tau;\mu,\phi;\mu',\phi')\text{"} \quad \text{for} \quad \frac{1}{4\pi}\int_\tau^\infty p(\tau';\mu,\phi;\mu',\phi')\cdot$$

$$\cdot \exp\left\{-(\tau'-\tau)\left[\frac{1}{\mu}-\frac{1}{\mu'}\right]\right\}d\tau' \quad . \tag{20}$$

If the space were separable, i.e., p were independent of τ (or, in the present case, the phase function component $\phi \equiv 0$) then writing "P_∞" for the limit of p as $\tau\to\infty$, we have:

$$P_\infty(\mu,\phi;\mu',\phi') = \frac{1}{4\pi}\frac{\mu\mu'}{\mu'-\mu}\,p_\infty(\mu,\phi;\mu',\phi') \quad , \tag{21}$$

where $(\mu,\phi)\ \varepsilon\ \Xi_+$, $(\mu',\phi')\ \varepsilon\ \Xi_-$. We observe that for eventually separable spaces,

$$\lim_{\tau\to\infty}\frac{dP(\tau;\mu,\phi;\mu',\phi')}{d\tau} = 0 \tag{22}$$

uniformly on $\Xi_+\times\Xi_-$, and that:

$$\lim_{\tau\to\infty}P(\tau;\mu,\phi;\mu',\phi') = P_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+\times\Xi_-$ and finally, that:

$$\lim_{\tau\to\infty}K^{j+1}(\tau,\mu,\phi) =$$

$$= \lim_{\tau\to\infty}\frac{\int_{\Xi_-}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')K^j(\tau,\mu',\phi')d\mu'd\phi'}{\int_{\Xi_-}P(\tau;\mu'\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi'}$$

(23)

for $(\mu,\phi)\ \varepsilon\ \Xi_+$. A similar expression holds for $K_q^{j+1}(\tau,\mu,\phi)$, which follows from (5) and (10):

$$\lim_{\tau\to\infty}K_q^{j+1}(\tau,\mu,\phi) =$$

$$= \lim_{\tau\to\infty}\frac{\int_{\Xi}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')K^j(\tau,\mu',\phi')d\mu'd\phi'}{\int_{\Xi}P(\tau;\mu,\phi;\mu',\phi')N^j(\tau,\mu',\phi')d\mu'd\phi'} \quad .$$

(24)

The next relation required below makes use of the forms of the principles of invariance in generally nonseparable media, in particular, use will be made of:*

$$N(\tau,\mu,\phi) = \frac{1}{\mu}\int_{\Xi_-} R(\tau,\infty;\mu,\phi;\mu',\phi')N(\tau,\mu',\phi')d\mu'd\phi'$$

where $(\mu,\phi) \varepsilon \Xi_+$ and $R(\tau,\infty;\cdot;\cdot)$ on $\Xi_+ \times \Xi_-$ is the reflectance function associated with the subset of $Z \times \Xi$ below level $\tau \geq 0$, i.e., with $X(\tau,\infty)$ (see (31) of Sec. 3.7). If the medium were separable, then for all pairs (τ_1,τ_2) of depths:

$$R(\tau_1,\infty;\mu,\phi;\mu',\phi') = R(\tau_2,\infty;\mu,\phi;\mu',\phi') \quad .$$

In the present case it is possible to verify on the basis of the differential equations for the R and T operators of Sec. 7.1 that:

$$\lim_{\tau\to\infty} \frac{dR(\tau,\infty;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

uniformly on $\Xi_+ \times \Xi_-$, and that:

$$\lim_{\tau\to\infty} R(\tau,\infty;\mu,\phi;\mu',\phi') = R_\infty(\mu,\phi;\mu',\phi')$$

uniformly on $\Xi_+ \times \Xi_-$, where R_∞ on $\Xi_+ \times \Xi_-$ is the reflectance of a homogeneous medium $X(0,\infty)$ with phase function p_∞. (R_∞ may be found using the methods of Sec. 7.6.)

Finally, the integral operator:

$$\int_{\Xi_-} [\quad] Q(\tau,\infty;\mu,\phi;\mu',\phi')d\mu'd\phi' \tag{26}$$

will be used. This operator maps the function $N(\tau,\cdot,\cdot)$ on Ξ_- into the function $N_q(\tau,\cdot,\cdot)$ on Ξ_-. The kernel Q is defined by writing:

"$Q(\tau,\infty;\mu,\phi;\mu',\phi')$" for $p(\tau;\mu,\phi;\mu',\phi')$

$$+ \int_{\Xi_+} p(\tau;\mu,\phi;\mu'',\phi'')R(\tau,\infty;\mu'',\phi'';\mu',\phi')\,\frac{d\mu''}{\mu''}\,d\phi''$$

---

*This section is adapted from Reference [224] with a minimum of notational change. Hence the reversal of the positions of N and R (and the primed and unprimed arguments) from that throughout the remainder of this work.

The operator (26) is a positive operator.* From the definition of Q, it follows once again from the differential equations for the R operators of Sec. 7.1 that:

$$\lim_{\tau\to\infty} \frac{dQ(\tau,w;\mu,\phi;\mu',\phi')}{d\tau} = 0$$

and that:

$$\lim_{\tau\to\infty} Q(\tau,\infty;\mu,\phi;\mu',\phi') = p_\infty(\mu,\phi;\mu',\phi') + \int_{\Xi_+} P_\infty(\mu,\phi;\mu'',\phi'')R_\infty(\mu'',\phi'';\mu',\phi')\frac{d\mu''}{\mu''}\,d\phi''$$

both uniformly on $\Xi_- \times \Xi_-$.

The Limit of $K_q(\cdot,\mu,\phi)$

We now begin the main steps of the proof. The object of the present discussion is to show that the function $K_q(\cdot,\cdot)$ on Ξ defined by writing:

$$\text{" } K_q(\mu,\phi) \text{ "} \quad \text{for} \quad \lim_{\tau\to\infty} K_q(\tau,\mu,\phi)$$

exists and is continuous almost everywhere on Ξ. The discussion begins with some observations on the functions K^j, K_q^j, $K_q^{(n)}$. In particular, we observe that for $(\mu,\phi)\in\Xi$ and every $\tau \geq 0$,

$$K_q^1(\tau,\mu,\phi) = \frac{\left\{\begin{aligned}&\int_{\Xi_-} p(\tau;\mu,\phi;\mu',\phi')N^0(\tau,\mu',\phi')\frac{d\mu'}{\mu'}\,d\phi' \\ &+\int_{\Xi_-} p'(\tau;\mu,\phi;\mu',\phi')N^0(\tau,\mu',\phi')\,d\mu'd\phi'\end{aligned}\right\}}{\int_{\Xi_-} p(\tau;\mu,\phi\ ,\mu',\phi')N^0(\tau,\mu',\phi')\,d\mu'd\phi'}$$

---

*For the present discussion, an operation T is said to be *positive* if T maps nonnegative functions into nonnegative functions and $Tf = 0$ (the zero function) implies f is the zero function, where f is a nonnegative valued function on Ξ_-, and the vanishing of f is taken in the sense of Lebesgue (cf., e ~., [111], p. 25).

where p' denotes the derivative of p with respect to τ. The function $N^o(\tau,\cdot,\cdot)$ on Ξ_- is related to the boundary radiance distribution by

$$N^o(\tau,\mu,\phi) = N^o(0,\mu,\phi)e^{\tau/\mu} \quad .$$

Hence each integrand in (27) is integrable on Ξ_-, so that $K_q^1(\tau,\mu,\phi)$ exists and is well defined for every $\tau \geq 0$ and $(\mu,\phi)\varepsilon\Xi$. Furthermore each integrand in (27) satisfies the hypothesis of Lebesgue's bounded convergence theorem, so that by (24):

$$K_q^1(\mu,\phi) = \lim_{\tau\to\infty} K_q^1(\tau,\mu,\phi) \tag{28}$$

exists for every $(\mu,\phi)\varepsilon\Xi$ and in fact $K_q^1(\cdot,\cdot)$ is continuous and therefore bounded on Ξ. The values of $K_q^1(\cdot,\cdot)$ are readily determinable for specific choices of $N^o(0,\cdot,\cdot)$. For example, if we adopt the standard discrete boundary radiance distribution defined by:

$$N^o(0,\mu,\phi) = N^o\delta(\mu-\mu_o)\delta(\phi-\phi_o) \quad , \quad -1 \leq \mu_o < 0 \quad ,$$

then:

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_o} \quad , \quad (\mu,\phi)\ \varepsilon\ \Xi \tag{29}$$

Slightly more generally, if:

$$N^o(0,\mu,\phi) = \sum_{i=o}^{n} N^o(\mu_i)\delta(\mu-\mu_i)\delta(\phi-\phi_i) \quad , \quad -1 \leq \mu_i < 0 \quad ,$$

and if:

$$\mu_o = \min\left\{ \mu_i : i = 0, 1, \ldots, n \right\}$$

then:

$$K_q^1(\mu,\phi) = -\frac{1}{\mu_o} \quad , \quad (\mu,\phi)\ \varepsilon\ \Xi \quad . \tag{30}$$

Other simple examples of $N^o(0,\cdot,\cdot)$ may be given, such as step-function representations, various simple continuous functions on Ξ_-, but (29) and (30) will suffice to illustrate the general procedure. In particular they help to illustrate the use of (15) which is required in the next step of the proof and which runs as follows: By means of (15) and (28), we see that for each $(\mu,\phi)\varepsilon\,\Xi_-$ such that

$$K_q^1(\mu,\phi) > -\frac{1}{\mu} \tag{31}$$

we have:

$$\lim_{\tau\to\infty} K^1(\tau,\mu,\phi) = -\frac{1}{\mu}$$

Since $K_q^1(\cdot,\cdot)$ is bounded, the subset of μ's for which (31) holds is a relatively open subset of $[-1,0]$ excluding 0. Finally, from (15) and (28), for each $(\mu,\phi)\in\Xi_-$ such that:

$$K_q^1(\mu,\phi) \le -\frac{1}{\mu} \quad , \tag{32}$$

we have:

$$\lim_{\tau\to\infty} K^1(\tau,\mu,\phi) = K_q^1(\mu,\phi) \quad .$$

Hence the function $K_\infty^1(\cdot,\cdot)$ on Ξ_- defined by writing:

$$"K_\infty^1(\mu,\phi)" \quad \text{for} \quad \lim_{\tau\to\infty} K^1(\tau,\mu,\phi)$$

exists for all $(\mu,\phi)\in\Xi_-$ and is continuous on Ξ_-. A particular illustration of a $K_\infty^1(\cdot,\cdot)$ is given by means of (29).

The main observation to make at this point is the following: In addition to being bounded on Ξ_-, the function $K_\infty^1(\cdot,\cdot)$ has the property that

$$K_\infty^1(\mu,\phi) \le -\frac{1}{\mu} \tag{33}$$

on Ξ_-. The discussion of $K_\infty^1(\cdot,\cdot)$ is completed by showing that it exists and is continuous on Ξ_+. This is done by applying the preceding arguments to (23). As an example, one may consider (29) once again, which yields $K_\infty^1(\mu,\phi) = -1/\mu_o$ for all $(\mu,\phi)\in\Xi_+$.

We now take the general inductive step, that is, we assume that $K_\infty^j(\cdot,\cdot)$, $j \ge 1$ is continuous on Ξ and in particular, $K_\infty^j(\mu,\phi) \le -1/\mu$ for every $(\mu,\phi)\in\Xi_-$. Then by means of (24) and the previously cited convergence arguments, we find that: $K_q^{j+1}(\mu,\phi)$, where we write:

$$"K_q^{j+1}(\mu,\phi)" \quad \text{for} \quad \lim_{\tau\to\infty} K_q^{j+1}(\tau,\mu,\phi) \tag{34}$$

exists for every $(\mu,\phi)\in\Xi_-$, and $K_q^{j+1}(\cdot,\cdot)$ is continuous on Ξ_-; and in particular,*

$$K_q^{j+1}(\mu,\phi) \le -\frac{1}{\mu}\ , \ (\mu,\phi)\in\Xi_- \quad . \tag{35}$$

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*To obtain (35), implicit use has been made of the general fact that if $F(x) = [A(z)a(z) + B(x)b(x)]/[A(x) + B(x)]$, and if we have $a(x) \to a(x_o) \le a_o$ along with $B(x) = o(A(x))$ as $x \to x_o$, then $F(x) \to a(x_o) \le a_o$, as $x \to x_o$.

Furthermore, by (15):

$$K_\infty^{j+1}(\mu,\phi) = \lim_{\tau\to\infty} K^{j+1}(\tau,\mu,\phi) = K_q^{j+1}(\mu,\phi) \tag{36}$$

on Ξ_-. Finally, from (23), $K_\infty^{j+1}(\cdot,\cdot)$ exists and is continuous on Ξ_+ and moreover,

$$K_\infty^{j+1}(\mu,\phi) \leq -\frac{1}{\mu}, \quad (\mu,\phi)\in\Xi_+ \quad . \tag{37}$$

Since the induction hypothesis has been demonstrated for the case $j+1$ assuming the case j, and it is true for $j = 1$, the conclusions (34)-(37) then hold for all integers $j = 1$, 2, ... It follows from (12) and the preceding results that:

$$K_q^{(n)}(\mu,\phi) = \lim_{\tau\to\infty} K_q^{(n)}(\tau,\mu,\phi) \leq -\frac{1}{\mu}, \tag{38}$$

exists and is continuous on Ξ, for $n = 2, 3, \ldots$.

Now consider the function $g_q^{(n)}(\cdot,\mu,\phi)$ on Z defined for every $(\mu,\phi)\in\Xi$ by writing:

$$"g_q^{(n)}(\tau,\mu,\phi)" \quad \text{for} \quad \frac{N_q^{(n)}(\tau,\mu,\phi)}{N_q(\tau,\mu,\phi)}, \quad n = 1, 2, \ldots . \tag{39}$$

Clearly $\{g_q^{(n)}(\cdot,\mu,\phi)\}$ is an increasing sequence of functions on Z, such that for every $(\mu,\phi)\in\Xi$,

$$\lim_{n\to\infty} g_q^{(n)}(\cdot,\mu,\phi) = 1 ,$$

the unit function on Z. From this and the definitions of $K_q^{(n)}$ and K_q, we conclude, first of all, that the sequence $\{K_q^{(n)}(\cdot,\mu,\phi)\}$ of functions converge in the mean to $K_q(\cdot,\mu,\phi)$ on Z. This in turn implies that the convergence to $K_q(\cdot,\mu,\phi)$ is almost uniform on Z for some subsequence $\{K_q^{(n_k)}(\cdot,\mu,\phi)\}$. Hence for every $\varepsilon > 0$, and subset Z_ε of Z,

$$\begin{aligned}\lim_{n_k\to\infty}\lim_{\tau\to\infty} K_q^{(n_k)}(\tau,\mu,\phi) &= \lim_{\tau\to\infty}\lim_{n_k\to\infty} K_q^{(n_k)}(\tau,\mu,\phi) \\ &= \lim_{\tau\to\infty} K(\tau,\mu,\phi)\end{aligned}$$

on $Z' = Z - Z_\varepsilon$, where

$$\int_{Z_\varepsilon} d\tau' < \varepsilon \quad .$$

It follows that the function $K_q(\cdot,\cdot)$ on Ξ has the property:

$$K_q(\mu,\phi) = \lim_{\tau\to\infty} K_q(\tau,\mu,\phi) = \lim_{n_k\to\infty} K_q^{(n_k)}(\mu,\phi) \leq -\frac{1}{\mu} \qquad (40)$$

for every (μ,ϕ) in Ξ. Finally, from (24), (15), (23), and (12) (in that order), we establish the fact that $\{K_q^{(n)}(\cdot,\cdot)\}$ is a sequence of continuous functions whose essential suprema form a nonincreasing sequence of real numbers. It follows that $K_q^{(n)}(\cdot,\cdot)$ converges uniformly a.e., on Ξ to $K_q(\cdot,\cdot)$, and that $K_q(\cdot,\cdot)$ is continuous on $\Xi - \Xi_o$ where Ξ_o is a subset of Ξ such that $\Omega(\Xi_o) = 0$. Hence $K_q(\cdot,\cdot)$ is continuous almost everywhere on Ξ.

The Limit of $K(\cdot,\mu,\phi)$

The proof is now concluded by showing that $K(\cdot,\mu,\phi)$ satisfies the definition of asymptoticity. By (14) and the result summarized in (40), we have

$$K_\infty(\mu,\phi) = \lim_{\tau\to\infty} K(\tau,\mu,\phi) = K_q(\mu,\phi) \qquad (41)$$

for all $(\mu,\phi) \in \Xi_-$. Hence by our preceding result on $K_q(\cdot,\cdot)$, $K_\infty(\cdot,\cdot)$ is continuous almost everywhere on Ξ_-. It follows [103, p. 242, problem (3)] that there is, for every $\varepsilon > 0$, a *compact* subset $\Xi_-(\varepsilon)$ of $\Xi_- - \Xi_o$ such that $\Omega[(\Xi_- - \Xi_o) - \Xi_-(\varepsilon)] < \varepsilon$ on which $K_\infty(\cdot,\cdot)$ is continuous. We use this fact to establish the existence of a minimal value of $K_\infty(\cdot,\cdot)$ on $\Xi_-(\varepsilon)$. Let (μ_1,ϕ_1) be any direction in $\Xi_-(\varepsilon)$ (there is at least one) defined by the condition:

$$K_\infty(\mu_1,\phi_1) = \inf\{K_\infty(\mu,\phi) : (\mu,\phi) \in \Xi_-(\varepsilon)\} \quad ,$$

and then write:

$$\text{"}g(\tau,\mu,\phi)\text{"} \quad \text{for} \quad \frac{N(\tau,\mu,\phi)}{N(\tau,\mu_1,\phi_1)} \quad ,$$

and observe that g on $\Xi_-(\varepsilon)$ defined by writing

$$\text{"}g(\mu,\phi)\text{"} \quad \text{for} \quad \lim_{\tau\to\infty} g(\tau,\mu,\phi)$$

is at least bounded and measurable (hence integrable) on $\Xi_-(\varepsilon)$. Then by means of the operator defined in (26), we have:

$$K_q(\mu,\phi) = \frac{\int_{\Xi_-(\varepsilon)} Q_\infty(\mu,\phi;\mu',\phi')g(\mu',\phi')K_\infty(\mu',\phi')d\mu'd\phi'}{\int_{\Xi_-(\varepsilon)} Q_\infty(\mu,\phi;\mu',\phi')g(\mu',\phi')d\mu'd\phi'} \quad , \qquad (42)$$

for all $(\mu,\phi)\varepsilon\,\Xi_-(\varepsilon)$. In particular, (41) holds for $(\mu_1,\phi_1)\varepsilon\,\Xi_-(\varepsilon)$. Using (41), (42) may be rewritten as:

$$\int_{\Xi_-(\varepsilon)} Q_\infty(\mu_1,\phi_1;\mu',\phi')g(\mu',\phi')[K_\infty(\mu',\phi') - K_\infty(\mu_1,\phi_1)]d\mu'd\phi' = 0$$

This operator $Q_\infty g$, as that in (26), being a positive operator, requires that the everywhere nonnegative valued function:

$$K_\infty(\cdot,\cdot) - K_\infty(\mu_1,\phi_1)$$

on $\Xi_-(\varepsilon)$ be the zero function almost everywhere on $\Xi_-(\varepsilon)$. Writing "k_∞" for $K_\infty(\mu_1,\phi_1)$ we have:

$$K_\infty(\mu,\phi) = k_\infty$$

for almost every $(\mu,\phi)\varepsilon\Xi_-(\varepsilon)$. Since ε is arbitrary, this result holds almost everywhere on $\Xi_- - \Xi_0$ and, by extension, everywhere on Ξ_- . An application of (25) to the definitions of $K(\tau,\mu,\phi)$ and $K_\infty(\mu,\phi)$, yields the result that

$$K_\infty(\mu,\phi) = k_\infty$$

for almost every (μ,ϕ) on Ξ_+ , so that $K_\infty(\cdot,\cdot)$ is a constant function almost everywhere (and, by our agreed extension, everywhere) on Ξ . *This concludes the proof.*

We observe finally, that, by means of the definition of N_q and (8),

$$K_q(\mu,\phi) = K_\infty(\mu,\phi) = k_\infty$$

everywhere on Ξ .

Notes and Observations

We now make an observation on the physical significance of the number k_∞ . We observe that the scalar irradiance function h on Z defined by writing

$$\text{"}h(\tau)\text{"} \quad \text{for} \quad \int_\Xi N(\tau,\mu,\phi)d\mu d\phi$$

has, in analogy to N , a K-function defined by writing:

$$\text{"}k(\tau)\text{"} \quad \text{for} \quad -\frac{1}{h(\tau)}\frac{dh(\tau)}{d\tau} \quad ,$$

which, as we saw in (39) of Sec. 9.2, is represented in terms of $K(\tau,\cdot,\cdot)$ by the formula:

$$k(\tau) = \frac{\int_{\Xi} N(\tau,\mu,\phi)K(\tau,\mu,\phi)\,d\mu d\phi}{\int_{\Xi} N(\tau,\mu,\phi)\,d\mu d\phi}$$

From this and the preceding results, we see that:

$$\lim_{\tau\to\infty} k(\tau) = \frac{\int_{\Xi} g(\mu,\phi)K_{\infty}(\mu,\phi)\,d\mu d\phi}{\int_{\Xi} g(\mu,\phi)\,d\mu d\phi} = k_{\infty} \quad .$$

Hence k_∞ is also the limit, as $\tau\to\infty$, of $k(\cdot)$, the K-function for scalar irradiance. The function h is related to the radiant energy density function u by $h = vu$, where v is the speed of light in the medium.

As a second observation, we note how the canonical form of the equation of transfer yields the integral equation for the asymptotic radiance distribution. The equation of transfer (3) may be written in terms of $K(\tau,\cdot,\cdot)$ as follows:

$$N(\tau,\mu,\phi) = \frac{N_q(\tau,\mu,\phi)}{1 + \mu K(\tau,\mu,\phi)}$$

which is the canonical form of the equation of transfer for the slab geometry (Chapter 4). The limit of the canonical form as $\tau\to\infty$ is, by the preceding results (and recall (16)):

$$\boxed{g(\mu,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} p_{\infty}(\mu,\phi;\mu',\phi')g(\mu',\phi')\,d\mu' d\phi'}{1 + \mu k_{\infty}}} \quad , \quad (43)$$

which is the general form used in the formal-solution procedures discussed in the introduction. The real number k_∞ now takes on the additional significance of being an eigenvalue of an eigenvalue problem associated with the above integral equation for g on Ξ. For the kind of boundary conditions adopted in the present section--which as we have noted before, stem from the geophysical origins of the asymptotic radiance problem--the resultant values of k_∞ are nonnegative, and in fact, $0 \leq k_\infty < 1$ (cf. (17) and the definition of g).

As a final observation, we relate the present theoretical findings to some independent computations and empirical measurements of asymptotic radiance distributions. Figure 10.13 shows the depth dependence of $K(\cdot,\mu,\phi)$ for several directions $(\mu,\phi)\varepsilon\Xi$. The associated medium is a hypothetical separable half-space, irradiated by normally incident collimated neutron flux, in which scattering is isotropic

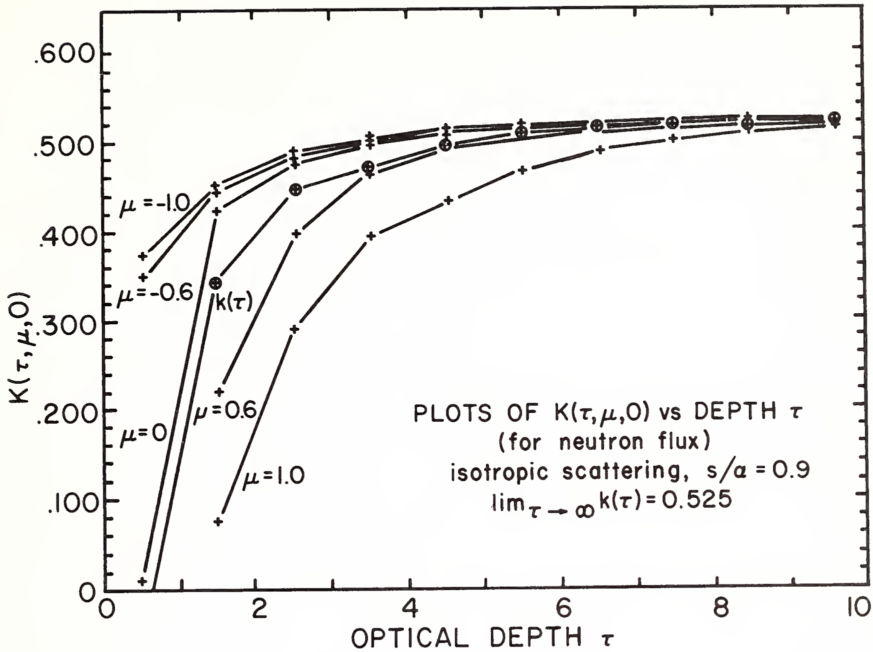

FIG. 10.13 A theoretical example of the asymptotic radiance theorem.

and $s/\alpha = 0.9$. These plots are based on theoretical computations of $N(\tau,\mu,\phi)$ (for neutron flux) compiled in [11]. The plots snow clearly that asymptoticity has been essentially attained at $\tau = 10$, for at this depth the function $K(10,\cdot,\cdot)$ is essentially constant on Ξ.

Figure 10.14 shows the depth dependence of $K(\cdot,\mu,\phi)$ for several directions $(\mu,\phi)\varepsilon\Xi$. The associated medium is a natural hydrosol, namely Lake Pend Oreille, Idaho, which at the time of measurement of $N(\tau,\mu,\phi)$, was irradiated by light from a clear sunny sky (angle of sun from zenith was about 40°, hence the associated μ_o was -0.77); scattering was found to be highly anisotropic and s/α approached, with increasing τ, a constant value of about 0.7, indicating the medium was eventually separable. These plots are based on experimental determinations of $N(\tau,\mu,\phi)$ recorded in [298]. All N-measurements were made at about 480 millimicrons. The plots show that asymptoticity is being markedly approached at depth $\tau = 20$ and below. The azimuth angle ϕ has been fixed at 0°, which denotes the vertical plane through the sun. Plots for $\phi \neq 0°$ indicate similar trends to asymptoticity for depths at $\tau = 20$ and below. The vertical K-scale has been exaggerated (relative to that of Fig. 10.13) in order to more clearly show the details of the transition to asymptoticity.

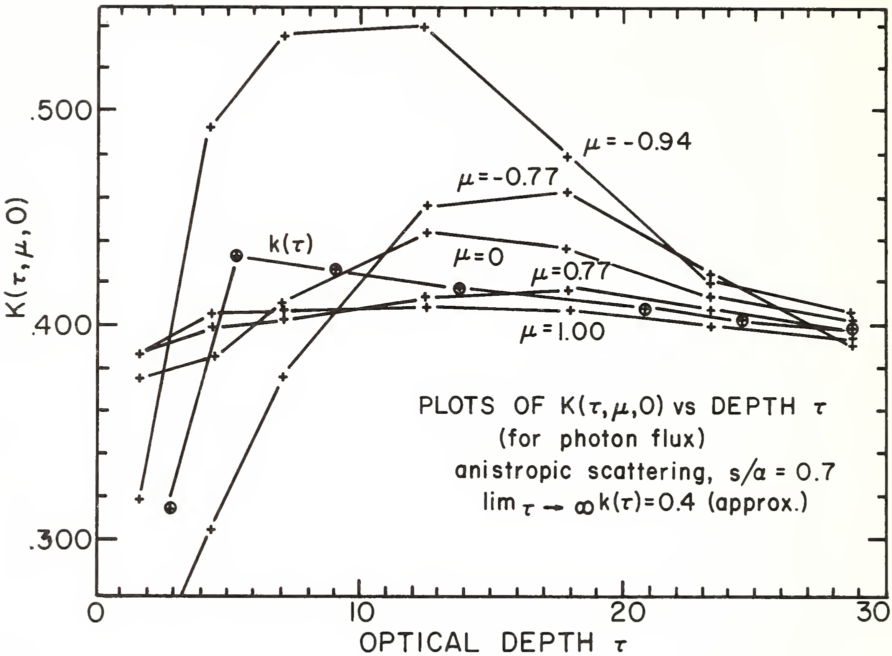

FIG. 10.14 An experimental example of the asymptotic radiance theorem.

10.6 On the Existence of Characteristic Diffuce Light: A Special Proof of the Asymptotic Radiance Hypothesis

In this section we return to the problem of the asymptotic radiance hypothesis and, as outlined in the introductory remarks to Sec. 10.5, we approach the hypothesis from a basically simpler, more empirical point of view. We shall therefore reintroduce the problem in the following paragraphs from this alternate point of view, and carry out the discussion so that it is virtually independent of that in Sec. 10.5.

Introduction

Recent experimental evidence, recorded in [298], forms the basis for fresh support of the long-standing conjecture that the radiance distribution about a point in an optically deep natural hydrosol approaches, with increasing depth, a characteristic form which is independent of the external lighting conditions and the optical state of the surface of the medium, and which depends only on the inherent optical properties of the medium. This conjecture was apparently given its first definitive formulation by Whitney [315], [316], who referred to the *asymptotic radiance distribution* as *characteristic diffuse light*. (We shall use these two

names interchangeably in what follows.) In this section we complement the experimental evidence in favor of this conjecture by supplying a simple proof of the existence of characteristic diffuse light in all homogeneous optically deep natural waters. The discussion concludes with a derivation of the integral equation governing the angular structure of the characteristic diffuse light and a brief discussion of an interesting and tractable example for the case of isotropic scattering.

We note in passing that since the time of the formulation of the asymptotic radiance hypothesis by Whitney, its domain of applicability has been widened considerably. The problem of a limiting angular distribution has since been encountered in modern neutron transport theory but basically as an abstract mathematical problem rather than experimental phenomenon. A similar type of problem has long been extant in astrophysical radiative transfer. A general proof of the existence of an asymptotic radiance distribution which covers all these contexts is given in Sec. 10.5.

Despite the widening of the domain of applicability of the hypothesis, it will retain its greatest usefulness in the context of geophysical optics, and in particular in hydrologic optics. For in this field, unlike the others mentioned above, the trend to a characteristic limiting form is a directly observable phenomenon. The existence of such a form is of inestimable importance to all experimental research work dealing with the determination of the optical properties of natural waters. In many important instances, knowledge that an asymptotic radiance distribution exists will obviate the necessity for experimental probings to extremely large depths; for such knowledge will allow, by means of relatively simple formulas, the accurate prediction of the geometrical structure of the light field in the great-depth ranges. Some of these practical consequences of the asymptotic radiance hypothesis are developed in Secs. 10.7 and 10.8.

Physical Background of the Method of Proof

The argument used by Whitney in establishing experimental evidence for the asymptotic radiance hypothesis went basically as follows: He showed that when the experimentally obtained plots of radiance distributions at various large depths were all blown up to the same size (more precisely, the zenith radiances were all normalized to a common value), they formed a set of nearly congruent figures. Now, an interesting feature of such radiance distributions is that they assume the same *shape*, and decrease in size with increasing depth at very nearly the same exponential rate. This fact can be stated precisely as follows:

$$N(z,\theta,\phi) = g(\theta,\phi)e^{-kz} \quad . \tag{1}$$

From this we see that the asymptotic radiance hypothesis is equivalent to the statement that *the directional and depth dependence of radiance distributions multiplicatively uncouple at great depths*. That is, the radiance function N

may be represented as the product of two functions: The function g gives the shape or directional structure common to all the distributions, and the exponential function gives the depth dependence of the distributions.

Each factor on the right hand side of (1) has special physical significance. The function g eventually defines the angular form of the characteristic diffuse light. The exponent k of the exponential function has the following interesting interpretation: We define the *scalar irradiance* $h(z)$ at depth z as usual by writing:

$$\text{"}h(z)\text{"} \quad \text{for} \quad \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} N(z,\theta,\phi) \sin\theta \, d\theta \, d\phi \quad . \tag{2}$$

The quantity $h(z)$ is a measure of the volume density of radiant energy at depth z. Measurements of $h(z)$ over the years in many hydrosols have shown that $h(z)$ varies essentially in an exponential manner with depth. That is, semi-log plots of $h(z)$ versus depth show an unmistakable trend toward linearity as depth increases. In any event, $h(z)$ may be accurately represented by a formula of the type:

$$h(z) = h(0) \exp\left\{ - \int_0^z k(z')dz' \right\} \quad , \tag{3}$$

where $k(z)$ is seen to be the logarithmic derivative of $h(z)$. As depth increases, the experimental evidence is that $k(z)$ approaches a constant value. Let us denote this limit value by "k_∞". Now assuming that an asymptotic radiance distribution is approached by the radiance distributions in a particular body of water, we see from (1), (2), and (3), that:

$$h(z) = h(z_o) e^{-k_\infty (z-z_o)} =$$

$$= e^{-kz} \int_{\phi=0}^{2\pi} \int_{\phi=0}^{\pi} g(\theta,\phi) \sin\theta \, d\theta \, d\phi \quad , \tag{4}$$

where z_o is the depth below which we may assume that $k(z) = k_\infty^o$. From this we conclude that:

$$k = k_\infty \tag{5}$$

Hence under the above assumption (1), we see that at great depths in the water, the size of the radiance distribution plots decrease exponentially with increasing depth, and the rate of this decrease is precisely that of the scalar irradiance (or radiant energy density).

The close connection between the depth dependence of scalar irradiance and that of the radiance distributions, as summarized in (5), suggests the following mode of representation of the radiance distributions for any depth: For each direction (θ,ϕ), we write as in (35) of Sec. 9.2:

$$"K(z,\theta,\phi)" \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} \quad . \tag{6}$$

Then, in analogy to (3), $N(z,\theta,\phi)$ at any depth z may be represented exactly by:

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^z K(z',\theta,\phi)\,dz'\right\} \quad . \tag{7}$$

Now suppose there is some depth z_o below which we have $K(z,\theta,\phi) = k_\infty$ for all directions (θ,ϕ). Then (7) may be written:

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^{z_o} K(z',\theta,\phi)\,dz' - \int_{z_o}^{z} K(z',\theta,\phi)\,dz'\right\}$$

$$= N(z_o,\theta,\phi)\exp\left\{-k_\infty(z - z_o)\right\}$$

If we write:

$$"g(\theta,\phi)" \quad \text{for} \quad N(z_o,\theta,\phi)\exp\left\{k_\infty z_o\right\} \quad ,$$

then we may go on to write:

$$N(z,\theta,\phi) = g(\theta,\phi)^{-k_\infty z} \quad , \tag{8}$$

for all depths z below z_o.

The similarity between (1) and (8) is unmistakable. This similarity points out the present method of attack we may follow in order to prove the asymptotic radiance hypothesis: We must show that the quantitites $K(z,\theta,\phi)$ approach a limit as depth is increased, and that this limit is independent of the directions (θ,ϕ). Furthermore, this limit, in accordance with the preceding discussion, should be none other than the limit k_∞ of $k(z)$, as defined in (3). Henceforth, we explicitly assume that k_∞, defined as the limit of $k(z)$ as $z\to\infty$, exists as a nonnegative number.

The Proof

We make use of the steady state source-free equation of transfer for radiance:

$$\frac{dN(z,\theta,\phi)}{dr} = -\alpha N(z,\theta,\phi) + N_*(z,\theta,\phi) \tag{9}$$

where, as usual:

$$N_*(z,\theta,\phi) = \int_{\phi'=o}^{2\pi} \int_{\theta'=o}^{\pi} N(z,\theta',\phi')\sigma(\theta',\phi';\theta,\phi)\sin\theta' d\theta' d\phi' \quad (10)$$

represents the path function N_*; σ is the volume scattering function (which governs the law of scattering in the water), and α is the volume attenuation coefficient. The formal solution of (9) is readily obtained and is simply the integral form of the equation of transfer (re: (6) of Sec. 3.13):

$$N(z,\theta,\phi) = N^o(z,\theta,\phi) + \int_o^r N_*(z',\theta,\phi)e^{-\alpha(r-r')}dr' \ . \quad (11)$$

The first term is the residual radiance which represents the component of N consisting of unscattered light. The second term is the path radiance which represents the space light over the path of length r, (Fig. 10.15). This path radiance has been generated by light scattered into the path of sight all along its extent. The formal solution (11) has been written for a general downward direction of flow of light (see Fig. 10.15), so that $N^o(z,\theta,\phi)$ is interpreted as the residual radiance transmitted from the upper boundary of the medium and is of the form:

$$N^o(z,\theta,\phi) = N^o(0,\theta,\phi)e^{-\alpha r}$$

where

$$-r\cos\theta = z \quad .$$

We now turn Equation (11) into a useful inequality by means of the following three steps:

First, since $N(z,\theta,\phi)$ clearly exceeds its path radiance component at all depths, we can write:

$$N(z,\theta,\phi) > \int_o^r N_*(z',\theta,\phi)e^{-\alpha(r-r')}dr' \quad .$$

Second, using the definition of N_*, we strengthen the inequality when we write:

$$N(z,\theta,\phi) > \sigma_{min} \int_o^r h(z')e^{-\alpha(r-r')}dr'$$

where σ_{min} is the minimum value of the volume scattering function; that is, we have used (10) to deduce that

$$N_*(z,\theta,\phi) > \sigma_{min} \int_{\phi'=o}^{2\pi} \int_{\theta'=o}^{\pi} N(z,\theta',\phi')\sin\theta' d\theta' d\theta' = \sigma_{min} h(z)$$

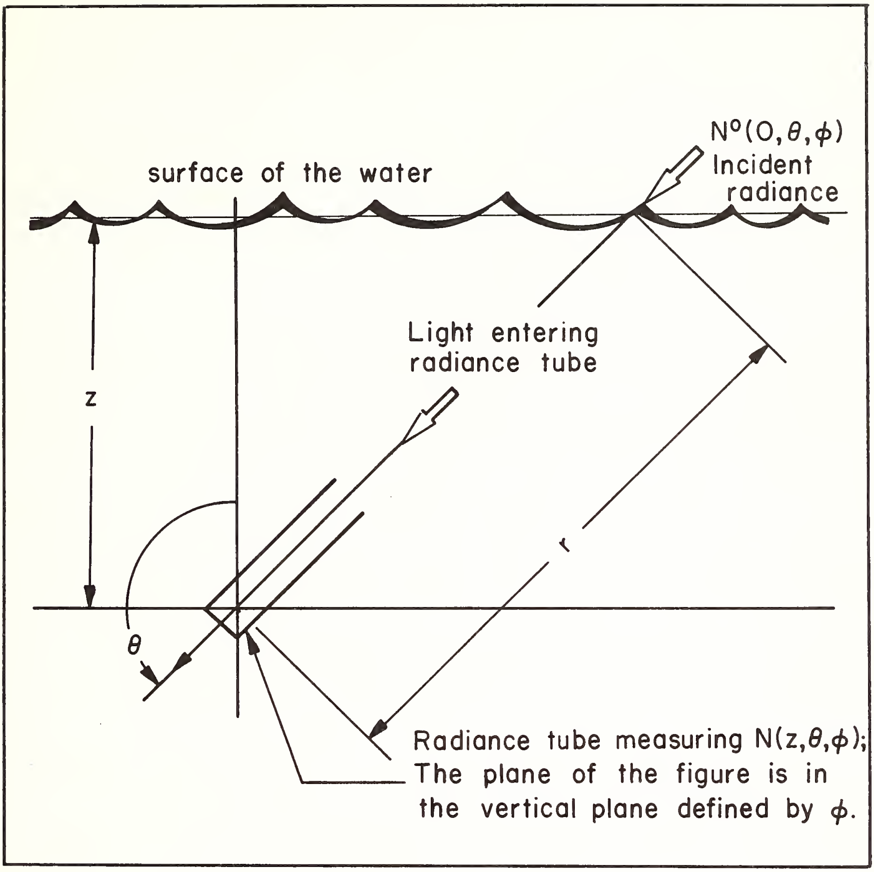

FIG. 10.15 Setting for a simplified proof of the asymptotic radiance theorem.

Finally, since h(z) generally decreases with increasing depth (i.e., k_∞ is positive), we certainly strengthen the inequality by writing:

$$N(z,\theta,\phi) > \sigma_{min} h(z) \int_0^r e^{-\alpha(r-r')} dr' \quad .$$

That is, we have:

$$N(z,\theta,\phi) > \frac{\sigma_{min}}{\alpha} h(z)(1 - e^{-\alpha r}) \tag{12}$$

for all depths z . From this we see that as depth z increases indefinitely, the exponential rate of decrease

$K(z,\theta,\phi)$ of the radiance cannot eventually exceed, and remain larger by any finite amount, the $k(z)$ of the scalar irradiance. For if it did, the plot of N would eventually fall and remain arbitrarily far below that of h. In other words, the ratio $N(z,\theta,\phi)/h(z)$ would go to zero, with increasing z, contrary to (12). The conclusion of this observation may be stated as follows:

$$\lim_{z\to\infty} K(z,\theta,\phi) \leq \lim_{z\to\infty} k(z) = k_\infty \tag{13}$$

for all downward directions (θ,ϕ). We now show that strict equality must hold in (13). We achieve this by initially assuming the contrary, that is, we assume that there is a set of directions Ξ_0 with positive solid angle measure over which:

$$\lim_{z\to\infty} K(z,\theta,\phi) \leq k_\infty - \varepsilon$$

where ε is an arbitrary small positive number. Then it is clear that the radiances in this set of directions decreases at a definitely smaller rate than the scalar irradiance, so much smaller, in fact that, by our assumption, it is true that for some depth z_1, we must have

$$\int_{\Xi_0} N(z_1,\theta,\phi)\sin\theta \, d\theta \, d\phi > h(z_1) \quad .$$

However, this conclusion clearly contradicts (2) (a part cannot exceed the whole). We have reached a contradiction which leaves only one other possibility, namely that:

$$\lim_{z\to\infty} K(z,\theta,\phi) = k_\infty \tag{14}$$

for all downward directions (θ,ϕ). In the light of the preceding discussions (cf. (8)) this means that the shapes of the radiance distributions impinging on the upper boundaries of deep layers of water eventually assume a fixed form. But it is known that the shape of the *reflected* radiance distribution at the upper boundary of a scattering layer is determined by the shape of the incident radiance distribution at that boundary (e.g., principle of invariance III of Example 3 of Sec. 3.7, with $N_+(b) = 0$ in the medium $X(0,\infty)$). Hence if the incident radiance distribution approaches a fixed shape, so does that of the reflected distribution. This completes the proof.

We observe that the present proof can also be applied in all natural waters which eventually become homogeneous. That is, the preceding arguments are basically unchanged if the medium is inhomogeneous over any initial finite depth range below the surface. Even more general situations exist which allow asymptotic radiance distributions, namely media in which the ratio σ/α eventually becomes independent of depth (Sec. 10.5).

The Equation for the Characteristic Diffuse Light

Using the equation of transfer, the definition (6), and the relation between z and r, we can write the equation of transfer in the following canonical form (Chapter 4):

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha + K(z,\theta,\phi)\cos\theta} \quad . \tag{15}$$

From (14) and (8) we see that the limiting form of (15) (as depth increases indefinitely) is

$$\boxed{g(\theta,\phi) = \frac{\displaystyle\int_{\phi'=0}^{2\pi}\int_{\theta'=0}^{\pi} g(\theta',\phi')\sigma(\theta',\phi';\theta,\phi)\sin\theta'\,d\theta'\,d\phi'}{\alpha + k_\infty\cos\theta}} \tag{16}$$

which is the equation governing the angular form of the characteristic diffuse light (cf. (43) of Sec. 10.5). It is a property of equations of the type shown in (16) that the function g is independent of ϕ for all real physical situations. Thus the characteristic diffuse light is always represented by a surface of revolution whose axis of symmetry is vertical.

The theory of the solution of such equations as (16) is fairly well understood (see e.g., [62]). The present discussions, therefore, will not consider in any detail the general solutions of (16). However, there is one simple special case which is immediately solved and which can shed much light on some of the salient details of the structure of the asymptotic radiance distributions. This is the case of isotropic scattering, where the volume scattering function σ is independent of direction and has the form:

$$\sigma(\theta,\phi;\theta',\phi') = \frac{s}{4\pi} \tag{17}$$

where s is the total scattering coefficient.

To see the resulting structure of the asymptotic radiance distribution, it is convenient in the present case to turn to (15). With the assumption (17) and the definitions (2) and (10), we have

$$N(z,\theta,\phi) = \frac{s}{4\pi}\,\frac{h(z)}{\alpha + K(z,\theta,\phi)\cos\theta} \quad ,$$

which at great depths approaches the form:

$$N(z,\theta,\phi) = \left(\frac{1}{4\pi}\right)\frac{s}{\alpha}\cdot\frac{h(z_0)e^{-k_\infty(z-z_0)}}{1 + \left(\frac{k_\infty}{\alpha}\right)\cos\theta} \quad . \tag{18}$$

Here z_o is the depth below which $h(z)$ is essentially of exponential behavior. Comparing (18) with (8), we see that for the present case,

$$\boxed{g(\theta,\phi) = \frac{1}{4\pi} \cdot \left(\frac{s}{\alpha}\right) \cdot \frac{h(z_o)e^{k_\infty z_o}}{1+\left(\frac{k_\infty}{\alpha}\right)\cos\theta}} \quad . \tag{19}$$

We have written (19) in the indicated form to point up the following geometric fact: A polar plot of $g(\theta,\phi)$ is generally a prolate ellipsoid of revolution with vertical axis, and of eccentricity k_∞/α. It is easy to deduce that when there is no absorption in the medium, then $k_\infty = 0$, and the characteristic diffuse light is represented by a sphere. On the other hand, if there is very little scattering as compared to absorption, the figure assumes a very narrow, pencil-like shape. In the limit of no scattering, k_∞ approaches α, and the figure degenerates into a vertical line.

The structure (18) is related to the limiting form for the simple canonical model for the apparent radiance (2) and (6) of Sec. 4.4 and is also related to a formula derived by Poole in Ref. [209]. We conclude with the observation that (19) predicts a different limiting ratio of the horizontal to the upward radiance than that derived by Whitney [316] under the same circumstances (i.e., isotropic scattering). Instead of the ratio 2:1, as suggested by Whitney, the present formula yields:

$$\frac{g(\pi/2,\phi)}{g(0,\theta)} = 1 + \left(\frac{k_\infty}{\alpha}\right) \le 2 \, . \tag{20}$$

In other words, the ratio in (20) is not a fixed magnitude, but depends on the optical properties of the medium in the manner shown.

The distribution (19) can serve as a convenient standard reference distribution against which experimentally determined radiance distribution can be compared. The amount of departure of the experimental distributions from (19) would then serve as a measure of the anisotropy of scattering in the real medium.

10.7 Some Practical Consequences of the Asymptotic Radiance Hypothesis

We shall now deduce some of the consequences of the asymptotic radiance hypothesis, as stated and proved in Secs. 10.5 and 10.6, for the case of the principal apparent optical properties of natural hydrosols.

It will be recalled that the asymptotic radiance hypothesis asserts that the angular distribution of radiance approaches a fixed form at great depths in eventually homogeneous natural waters. We shall show below that the

following consequences can be deduced from the asymptotic radiance hypothesis: The logarithmic derivatives (with respect to depth z) of radiance values $N(z,\theta,\phi)$ approach, with increasing z, a common fixed value k_∞ for all directions (θ,ϕ); further, the logarithmic derivatives of scalar irradiance $h(z)$, its upwelling and downwelling components $h(z,+)$ and $h(z,-)$, along with the derivatives of the upwelling and downwelling irradiances $H(z,+)$ and $H(z,-)$ all approach the common limit k_∞ as depth increases. Further consequences are that the two-D model for the irradiance field in natural waters (Sec. 8.5) becomes exact with increasing depth. These and related results are illustrated by examples drawn from the special case of isotropic scattering. Finally, a formula is developed which allows an estimate of the depth at and below which the actual radiance distributions differ from the asymptotic distribution by no more than a preassigned amount. Thus, the formula may constitute a criterion for asymptoticity in natural hydrosols.

In order to keep the present discussion essentially self contained and useful for references purposes, we shall review the definitions and properties of the various concepts to which the asymptotic radiance hypothesis will be applied.

Basic Formulas: The Irradiance Quartet

As is demonstrated in Chapter 2, the radiance function N is a basic radiometric quantity in terms of which all others can be defined. In particular the downwelling and upwelling irradiances $H(z,-)$ and $H(z,+)$ at depth z in a natural hydrosol are given by:

$$H(z,-) = - \int_{\Xi_-} N(z,\theta,\phi) \cos\theta \, d\Omega \quad , \qquad (1)$$

and

$$H(z,+) = \int_{\Xi_+} N(z,\theta,\phi) \cos\theta \, d\Omega \quad , \qquad (2)$$

where, as usual, we define Ξ_- as the collection of all downward (or inward) directions (θ,ϕ): $\pi/2 < \theta$, $0 \le \phi < 2\pi$, and Ξ_+ is the collection of all upward (or outward) directions (θ,ϕ): $\theta < \pi/2$, $0 \le \phi < 2\pi$, where θ is measured as usual from the outward normal $\mathbf{k}$ to the medium. We have thus partitioned the set Ξ of all directions into its *upper* (+) *hemisphere* and its *lower* (-) *hemisphere*. For brevity we have written, "$d\Omega$" for $\sin\theta \, d\theta \, d\phi$, in (1) and (2).

In addition to $H(z,+)$ and $H(z,-)$, underwater optical experiments usually consider the following downwelling and upwelling scalar irradiances:

$$h(z,-) = \int_{\Xi_-} N(z,\theta,\phi)\,d\Omega \quad , \tag{3}$$

and

$$h(z,+) = \int_{\Xi_+} N(z,\theta,\phi)\,d\Omega \quad , \tag{4}$$

and their sum $h(z)$:

$$h(z) = h(z,-) + H(z,+) \quad , \tag{5}$$

which is the scalar irradiance at depth z ($h(z)$ is equal to the product of the speed of light v and radiant density $u(z)$ at depth z). These concepts were introduced and discussed in detail in Chapter 2.

The four quantities $H(z,\pm)$, $h(z,\pm)$, so useful in the formulation of the two-flow theory of Chapter 8, also form the nucleus of a set of modern *experimental quantities* used to document the light field in natural waters. Of course, a complete documentation is obtained only through a systematic determination of the radiance values $N(z,\theta,\phi)$ at all depths z and over all directions (θ,ϕ). Nevertheless, as is seen in the developments of Chapters 8 and 9, this quartet of irradiances contains an extraordinary amount of useful radiometric information compactly packaged.

The D and R Functions

In the absence of detailed knowledge of $N(z,\theta,\phi)$ the basic quartet of irradiance functions defined above can be used to derive most of the information needed for the solution of underwater visibility problems, and image and flux transmission problems in general. In particular, an excellent index of the shape of the radiance distributions at depth z is given by the distribution functions $D(z,\pm)$ represented as:

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} \quad . \tag{6}$$

Furthermore, information about the reflectance properties of the water at depth z is furnished by a study of the ratio:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad , \tag{7}$$

which is the experimental counterpart to the classical R_∞ formula as given by classical one-dimensional two-flow analysis of the light field. In fact, the D and R functions defined above, and the functions defined below are all either modern experimental counterparts or logical extensions of the

tools provided by the classical two-flow theory of the light field in natural hydrosols. As noted above, the background of these particular radiometric quantities is considered in detail in Chapters 8 and 9 so that the present discussion need not dwell further on their definitions and interrelations. We are concerned here only with the behavior of these quantities at great depths in media satisfying the requirement of the asymptotic radiance hypothesis.

The K Functions

The essentially exponential behavior of the irradiance quantities supplies the motivation for the definitions of the K functions (Chapter 9) assembled here for convenience:

$$K(z,\pm) = -\frac{1}{H(z,\pm)}\frac{dH(z,\pm)}{dz} , \tag{8}$$

$$k(z,\pm) = -\frac{1}{h(z,\pm)}\frac{dh(z,\pm)}{dz} , \tag{9}$$

$$k(z) = -\frac{1}{h(z)}\frac{dh(z)}{dz} . \tag{10}$$

If the various irradiance quantities vary *exactly* in an exponential manner at *all* depths, then the corresponding K functions would be constant functions each assuming a fixed value at all depths. In general, however, (Secs, 10.1-10.4) the depth-dependence of these quantities is nonconstant and it is only after 10-20 attenuation lengths that the exponential features eventually emerge. The preceding representations, however, are designed to characterize the depth-dependence of the irradiances under all conditions.

One of the main consequences derived from the asumptotic radiance hypothesis is that the five K functions represented above all tend to a common limit with increasing depth. We prepare the groundwork leading to this conclusion by reintroducing the K function for the radiance function itself (re (34) of Sec. 9.2): Thus we write:

$$\text{"}K(z,\theta,\phi)\text{"} \quad \text{for} \quad -\frac{1}{N(z,\theta,\phi)}\frac{dN(z,\theta,\phi)}{dz} . \tag{11}$$

Just as each of the various irradiance quantitites may be expressed in terms of radiance, so can its corresponding K function be expressed in terms of the K function for radiance:

$$K(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)K(z,\theta,\phi)\cos\theta\, d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi)\cos\theta\, d\Omega} , \tag{12}$$

$$k(z,\pm) = \frac{\int_{\Xi_\pm} N(z,\theta,\phi)K(z,\theta,\phi)\,d\Omega}{\int_{\Xi_\pm} N(z,\theta,\phi)\,d\Omega} \,, \tag{13}$$

$$k(z) = \frac{\int_{\Xi} N(z,\theta,\phi)K(z,\theta,\phi)\,d\Omega}{\int_{\Xi} N(z,\theta,\phi)\,d\Omega} \,. \tag{14}$$

The K Characterization of the Hypothesis

The K function $K(z,\theta,\phi)$ for radiance is of fundamental importance in the present discussion of the asymptotic radiance hypothesis. In fact it is the function which gives rise to that form of the hypothesis which is most amenable to exact mathematical analysis. The desired characterization reads as follows ((16) of Sec. 10.5) *for each $(\theta,\phi)\varepsilon\,\Xi$, the function $K(z,\theta,\phi)$ has a limit, as $z \to \infty$, and this limit is independent of (θ,ϕ).* In symbols:

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

exists for every $(\theta,\phi)\varepsilon\,\Xi$, and is independent of (θ,ϕ).

The preceding formulation, as explained in Secs. 10.5 and 10.6, is made plausible by the following observations: For every depth z, $N(z,\theta,\phi)$ may be represented exactly by

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^z K(z',\theta,\phi)\,dz'\right\} \,.$$

Suppose there is some depth z_o below which we have $K(z,\theta,\phi) = k_\infty$, a fixed number for all (θ,ϕ). Then

$$N(z,\theta,\phi) = N(0,\theta,\phi)\exp\left\{-\int_0^{z_o} K(z',\theta,\phi)\,dz' - \int_{z_o}^{z} K(z',\theta,\phi)\,dz'\right\}$$

$$= N(z_o,\theta,\phi)\exp\left\{-k_\infty(z-z_o)\right\} \,.$$

Write

$$\text{"}g(z_o,\theta,\phi)\text{"} \quad \text{for} \quad N(z_o,\theta,\phi)\exp\left\{k_\infty z_o\right\} \,,$$

then for all $z \geq z_o$,

$$N(z,\theta,\phi) = g(z_o,\theta,\phi)e^{-k_\infty z} \quad . \tag{15}$$

It follows that below z_o , $N(z,\theta,\phi)$ has a fixed angular structure given by $g(z_o,\theta,\phi)$.

The Basic Transfer Equations

One final relation needed below is the reformulation of the equation of transfer in terms of the K function for radiance. This is easily obtained from the standard form of the transfer equation for stratified source-free plane-parallel media:

$$-\cos\theta \frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad ,$$

where:

$$N_*(z,\theta,\phi) = \int_\Xi N(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)d\Omega \quad .$$

By means of the definition of $K(z,\theta,\phi)$, the above equation may be rewritten in its canonical form (Chapter 4):

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha(z) + K(z,\theta,\phi)\cos\theta} \quad , \tag{16}$$

The equation of transfer governing $K(z,\theta,\phi)$ is also easily found. From (16), the definition of $K(z,\theta,\phi)$, and the following definition of an analogous K function:

$$K_q(z,\theta,\phi) = -\frac{1}{N_q(z,\theta,\phi)} \frac{dN_q(z,\theta,\phi)}{dz} \quad , \tag{17}$$

where

$$N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z) \quad , \tag{18}$$

we have:

$$\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - K_q(z,\theta,\phi)][K(z,\theta,\phi) + \alpha(z)\sec\theta] \tag{19}$$

and which played a key role in the proof arguments of Sec. 10.5.

This formulation is analogous to the following formulation of the equation of transfer for $N(z,\theta,\phi)$ in which $N_q(z,\theta,\phi)$ is used:

$$\frac{dN(z,\theta,\phi)}{dz} = [N(z,\theta,\phi) - N_q(z,\theta,\phi)][+ \alpha(z)\sec\theta)] \quad .$$

These formulations point up the following physical significance of the equilibrium radiance N_q and its K function K_q: for $\theta > (\pi/2)$ we observe that if $N(z,\theta,\phi) \lessgtr N_q(z,\theta,\phi)$ then $dN(z,\theta,\phi)/dz \gtrless 0$. This follows immediately from the preceding equation. Thus N, $\theta > \pi/2$, always tends toward the equilibrium radiance N_q. Now a similar phenomenon exists between K and K_q. To see this, we observe that the second factor on the right in (19) has the property that:

$$K(z,\theta,\phi) + \alpha(z)\sec\theta < 0 \quad ,$$

for all z and all downward directions (θ,ϕ). Therefore if $K(z,\theta,\phi) \lessgtr K_q(z,\theta,\phi)$ then $dK(z,\theta,\phi)/dz \gtrless 0$, showing that K always tends toward K_q for these directions. This property of the function $K(z,\theta,\phi)$ provided the key to the rigorous proof of the existence of an asymptotic radiance distribution given in Sec. 10.5. A further example of such a use of (19) is given in the final sections below.

Consequences for Directly Observable Quantities: The Equation for the Asymptotic Radiance Distribution

An application of the asymptotic radiance hypothesis to (16) yields the formula for the asymptotic radiance distribution g. In view of the heuristic discussion leading to (15) and the statement of the hypothesis in terms of $K(z,\theta,\phi)$, we see that

$$\lim_{z\to\infty} g(z,\theta,\phi) = \lim_{z\to\infty} N(z,\theta,\phi)\exp\left\{k_\infty z\right\}$$

exists for all (θ,ϕ). We shall denote this limit by "$g(\theta,\phi)$". Hence multiplying each side of (16) by $\exp\{k_\infty z\}$ and passing to the limit as $z\to\infty$, we have:

$$g(\theta,\phi) = \frac{\frac{1}{4\pi}\int_{\Xi} g(\theta',\phi')p(\theta',\phi';\theta,\phi)\,d\Omega}{1 + \left(\frac{k_\infty}{\alpha}\right)\cos\theta} \qquad (20)$$

where:

$$p(\theta',\phi';\theta,\phi) = \lim_{z\to\infty} 4\pi\sigma(z;\theta',\phi';\theta,\phi)/\alpha(z)$$

and:

$$k_\infty = \lim_{z\to\infty} K(z,\theta,\phi)$$

and*

$$\alpha = \lim_{z\to\infty}\alpha(z) \quad .$$

The integral equation (20) has the property that the values of its solution g are independent of θ. Thus we may write:

$$"g_o(\theta)" \quad \text{for} \quad 2\pi g(\theta,\phi)$$

and (20) may be simplified to read:

$$\boxed{g_o(\theta) = \frac{\frac{1}{2}\int_{\theta'=0}^{\pi} g_o(\theta')p^{(o)}(\theta';\theta)\sin\theta'\,d\theta'}{1 + \left(\frac{k_\infty}{\alpha}\right)\cos\theta}} \quad , \qquad (21)$$

where we have written:

$$"p^{(o)}(\theta';\theta)" \quad \text{for} \quad \frac{1}{2\pi}\int_{\theta'=0}^{2\pi} p(\theta',\phi';\theta,\phi)\,d\phi' \quad .$$

The function g_o describes the essential geometric form of the asymptotic radiance distribution. A graph of g_o is clearly a surface of revolution with vertical axis (in the coordinate system of the plane-parallel medium). Furthermore, the structure of g_o and the value of k_∞ are completely determined by the phase function p (k_∞/α plays the role of an eigenvalue of the equation (21)). Thus g_o is determined completely by the inherent optical properties of the medium by means of equation (21) and therefore is independent of the external lighting conditions. The extension of (21) to the polarized case is immediate, upon starting the preceding derivation with the canonical equation of transfer (8) of Sec. 4.6 for the standard observable radiance vector.

The Limits of the K Functions

From the relations (12)-(14), and the statement of the hypothesis, we conclude that:

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*For most practical situations, the medium is homogeneous or eventually homogeneous, so that this limit exists. Actually, as shown in Sec. 10.5., the asymptotic radiance distribution exists whenever $\lim_{z\to\infty}\sigma/\alpha$ exists, without necessarily requiring that the individual limits $\lim_{z\to\infty}\sigma$ and $\lim_{z\to\infty}\alpha$ exist.

$$\lim_{z\to\infty} K(z,\pm) = k_\infty \quad , \tag{22}$$

$$\lim_{z\to\infty} k(z,\pm) = k_\infty \quad , \tag{23}$$

$$\lim_{z\to\infty} k(z) = k_\infty \quad . \tag{24}$$

The limit (24) is interpreted as follows: The logarithmic derivative (with respect to z) of scalar irradiance $h(z)$ eventually approaches the common limit k of the logarithmic derivatives of radiance distribution $N(z,\theta,\phi)$. The limit (23) shows that the logarithmic derivatives of the upwelling and downwelling irradiances (which as we saw in Sec. 10.2, are measurably distinct at all small depths z) approach a common value, namely k_∞. A similar interpretation holds for the K functions of the upwelling and downwelling scalar irradiances.

The Limits of the D and R Functions

From (6) and the hypothesis, we have immediately:

$$\lim_{z\to\infty} D(z,\pm) = \frac{\int_\pm g_o(\theta)\sin\theta\, d\theta}{\int_\pm g_o(\theta)[\pm\cos\theta]\sin\theta\, d\theta} \quad , \tag{25}$$

which we shall denote by "$D(\pm)$". Furthermore in (25) we have written:

$$\text{"}\int_+\text{"} \text{ for } \int_{\theta=0}^{\pi/2} \quad , \quad \text{and} \quad \text{"}\int_-\text{"} \text{ for } \int_{\theta=\pi/2}^{\pi} \quad .$$

In (25) of Sec. 9.2 it was shown that $R(z,-)$ can be represented quite generally in terms of the K functions and the distribution functions as follows:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)} \quad ,$$

where:

$$a(z,\pm) = D(z,\pm)a(z) \quad ,$$

and where $a(z)$ is the value of the volume absorption function of the medium at depth z. Let us write:

$$\text{"}R_\infty\text{"} \quad \text{for} \quad \lim_{z\to\infty} R(z,-)$$

It follows from the preceding representation for $R(z,-)$ that R_∞ exists and is given by:

$$\boxed{R_\infty = \frac{k_\infty - a(-)}{k_\infty + a(+)}} \quad , \qquad (26)$$

where:

$$\lim_{z\to\infty} a(z,\pm) = \lim_{z\to\infty} D(z,\pm)a(z) = D(\pm)a \quad .$$

and where we have written:

$$\text{"}a(\pm)\text{"} \quad \text{for} \quad D(\pm)a \quad .$$

Further limit relations may be determined by systematically going through the set of directly observable quantities discussed in Chapter 9. The preceding results will serve to illustrate the general procedure of obtaining the desired limit expressions.

We observe that (26) is similar to the classical expression for R_∞ as given by the two-D model for the irradiance field ((102) of Sec. 8.7). This similarity is not coincidental; it is, rather a consequence of the fact that under the asymptotic radiance hypothesis, the general two-D model becomes exact with increasing depth. We now consider this fact in more detail.

Consequences for Some Simple Theoretical Models: The Two-D Model for Irradiance Fields

In Chapter 8 a study of the classical two-flow equations for $H(z,+)$ and $H(z.-)$ showed that these equations were exact if and only if the distribution functions $D(z,+)$ and $D(z,-)$ were independent of depth (Sec. 8.5). Under the asymptotic radiance hypothesis it was seen in (25) that the distribution functions become independent of depth at great depths. It follows that the two-D equations for undecomposed irradiance $H(z,+)$ and $H(z,-)$ become exact at great depths whenever the hypothesis holds.

In Chapter 8 a formulation of the equations for $H^*(z,+)$ and $H^*(z,-)$ (the decomposed irradiances associated with diffuse light) was made in which each stream of flux was assigned a fixed distribution factor $D^*(+)$, $D^*(-)$ (the two-D theory for decomposed irradiance). This formulation was justified on the basis of experimental evidence which showed that $D(z,+)$ and $D(z,-)$ were essentially fixed (generally distinct) constants. In the light of the present analysis, the use of the two-D theory for decomposed irradiance is also given further justification on theoretical grounds whenever the asymptotic radiance hypothesis holds.

The two-D model gives explicit formulas for $H^*(z,+)$ and $H^*(z,-)$. In view of the preceding observations, these

expressions become exact with increasing depth z. Using the equations developed in Sec. 8.6 it may be shown that for every depth z in an infinitely deep medium $X(0,\infty)$:

$$H^*(z,-) = N^o C(\mu_o,-) \left[e^{-k_\infty z} - e^{-\alpha z/\mu_o} \right] , \quad (27)$$

$$H^*(z,+) = N^o \left[C(\mu_o,-) \frac{g_-(+)}{g_+(-)} e^{-k_\infty z} - C(\mu_o,+) e^{-\alpha z/\mu_o} \right] . \quad (28)$$

Observe that we have set $k_\infty = -k_-$, where k_- is given in (12) of Sec. 8.5. The physical setting associated with (27) and (28) is an infinitely deep plane-parallel slab irradiated by collimated flux incident at the upper boundary at an angle $\theta_o = \arccos \mu_o$ from the outward unit normal. The response to an arbitrary incident distribution is obtained by integrating (27) and (28) over Ξ_- (see Sec. 10.3). $C(\mu_o,\pm)$ are constants for each μ_o, determined by the optical parameters and boundary conditions; and:

$$g_-(\pm) = 1 \mp \frac{a(\mp)}{k_\infty} ,$$

where $a(\pm)$ are as defined in (26). The observable irradiances $H(z,\pm)$ are, by definition,

$$H(z,\pm) = H^o(z,\pm) + H^*(z,\pm) ,$$

where:

$$H^o(z,+) = 0 ,$$

$$H^o(z,-) = N^o \mu_o e^{-\alpha z/\mu_o} .$$

The expressions for the observable irradiances are given in (1) and (2) of Sec. 10.3. The preceding model yields the following prediction of the limit of $R(z,-)$:

$$R_\infty = \lim_{z\to\infty} R(z,-) = \lim_{z\to\infty} \frac{H(z,+)}{H(z,-)} = \frac{g_-(+)}{g_-(-)} = \frac{k_\infty - a(-)}{k_\infty + a(+)} ,$$

which agrees with (26), the exact limit given by general radiative transfer theory. These observations show that in any medium in which the asymptotic radiance hypothesis holds, if discussions are restricted to the class of all possible *two-flow* models of the light field, the model which attains maximal accuracy is that given by the two-D theory.

Critique of Whitney's "General Law"

After conjecturing that the radiance distributions assume a fixed shape at great depths, L. V. Whitney made use of the conjecture to deduce a so-called "general law of the

diminution of light intensity in natural waters" (cf. ref. [316]). An examination of the differential equations formulating this law reveals that they are incomplete: They fail to account for the contribution to the downwelling irradiance by the backscattered fraction of the upwelling irradiance. As a result, the solutions of the differential equations are generally inadequate to cope with the contribution from one half of the light field, namely the component associated with the upwelling flux. Furthermore, some (convenient, but incorrect) assumptions were made about the depth rate of change of the mean free path for unscattered light at various depths. On this basis the equations were integrated, holding fixed the mean free path for directly transmitted light. Both of these inadequacies of an otherwise satisfactory theory have been remedied in the two-D theory of the light field. The equations (27) and (28) (or their observable counterparts (1) and (2) of Sec. 10.3) represent the concomitant effects of both upwelling and downwelling streams. Finally, the awkwardness stemming from the change with depth of the mean free path of directly transmitted light has been avoided by considering only collimated incident flux of radiance N^o at the upper boundary.

The Simple Model for Radiance Distributions

In (2) of Sec. 4.4 a simple model for radiance distributions was derived in the form of the canonical representation of the apparent radiance. One key assumption in the establishment of the model was the depth-independence of the K function for radiance. In view of the preceding observations, it is concluded that the simple canonical model becomes exact with increasing depth in all media in which the asymptotic radiance hypothesis holds.

Further Consequences of Asymptoticity

We conclude with some examples drawn from the case of a plane-parallel medium which exhibits isotropic scattering and in which the asymptotic radiance hypothesis holds. In this way we obtain some general ideas about the shape of g_o, as governed by (21), and the order of magnitudes of the quantities $D(\pm)$, R_∞, and k_∞ one may expect in *real* media. Finally, it is possible to give, in the present context, a simple heuristic proof of the hypothesis, and at the same time derive a formula which will provide a means of determining the depth in a medium below which asymptoticity has essentially been attained. We shall now consider these matters in turn.

The Standard Ellipsoid

When scattering is isotropic, the phase function takes on the form:

$$p(\theta',\phi';\theta,\phi) = \rho = s/\alpha \quad ,$$

where s is the volume total scattering coefficient and "ρ" is an abbreviation of "s/α", and both denote the scattering-attenuation ratio. Using this phase function in (21) we see that $g_o(\theta)$ takes on a particularly simple form,

$$g_o(\theta) = \frac{\rho}{2} \cdot \frac{g_o}{1 + \varepsilon \cos\theta} \tag{29}$$

where we have written

$$\text{"}\varepsilon\text{"} \quad \text{for} \quad k_\infty/\alpha \quad ,$$

and

$$\text{"}g_o\text{"} \quad \text{for} \quad \int_{\theta=0}^{\pi} g_o(\theta)\sin\theta\, d\theta \quad .$$

Physical significance can be attached to g_o by returning to the definition of $g(\theta,\phi)$ and integrating over Ξ (see (15) and (20)). The result is:

$$g_o = \lim_{z\to\infty} h(z)e^{k_\infty z} \quad .$$

Hence if there is some depth z_o below which one may consider that for practical purposes asymptoticity has been attained, then the preceding relation can be written:

$$g_o = h(z_o)e^{k_\infty z_o} \quad .$$

Expression (29) represents a prolate spheroid of revolution whose axis of symmetry is vertical. The eccentricity of the ellipsoid is $\varepsilon = k_\infty/\alpha$. This ellipsoid may serve as a convenient reference against which distributions from real media may be compared. To effect a comparison one must know the ρ and ε of the medium. Since ε and ρ are generally related, it suffices in principle to know only ρ and the phase function. This is illustrated below after a necessary preliminary discussion of $D(\pm)$ and R_∞.

Expressions for $D(\pm)$ and R_∞

By means of (25) and (29) we find that (see also (57) and (58) of Sec. 2.11):

$$D(\pm) = \frac{\varepsilon \ln(1 \pm \varepsilon)}{\varepsilon \mp \ln(1 \pm \varepsilon)} \quad . \tag{30}$$

Furthermore, from (7) and (29)(i.e., (29) replaces $N(z,\theta,\phi)$ in (7)), we have:

$$R_\infty = \frac{\ln(1 + \varepsilon) - \varepsilon}{\ln(1 - \varepsilon) + \varepsilon} \quad . \tag{31}$$

The same result could be obtained by using (26) and the preceding form for $D(\pm)$.

Values of $D(\pm)$ and R_∞ as functions of ε, $0 < \varepsilon < 1$ are given in Table 1. It is easy to verify that for the extreme values 0 and 1 of ε the corresponding values of $D(\pm)$ and R_∞ are:

$$\lim_{\varepsilon\to 0} D(\pm) = 2 \quad ;$$

$$\lim_{\varepsilon\to 1} D(+) = \frac{\ln 2}{1 - \ln 2} = 2.259 \quad ;$$

$$\lim_{\varepsilon\to 1} D(-) = 1 \quad .$$

$$\lim_{\varepsilon\to 0} R_\infty = 1 \quad ,$$

$$\lim_{\varepsilon\to 1} R_\infty = 0 \quad .$$

Table 1 of Sec. 8.5 gives values of $D(z,\pm)$ for a real medium under varying external conditions. A comparison of these real values with those summarized in Table 1 below reveals the following information: The $D(z,-)$ values are significantly less than the standard $D(-)$ values; the $D(z,+)$ values are significantly greater than the standard $D(+)$ values. Since all natural waters exhibit anisotropic scattering we can infer the following features of the structure of asymptotic radiance distributions in all natural waters: When compared with the standard ellipsoid, the plots of $g_o(\theta)$ for real media must necessarily be narrower in the angular range $\theta > \pi/2$ (upwelling light).

The amount of departure of the $g_o(\theta)$ for a real medium from the standard ellipsoid may be taken as a measure of the anisotropy of scattering in the medium.

TABLE 1

Distribution and reflectance factors for standard ellipsoid

ε	D(-)	D(+)	R_∞
0.100	1.9664	2.0319	0.8750
0.200	1.9286	2.0622	0.7640
0.300	1.8881	2.0911	0.6642
0.400	1.8438	2.1185	0.5733
0.500	1.7943	2.1143	0.4895
0.600	1.7381	2.1692	0.4110
0.700	1.6722	2.1928	0.3361
0.800	1.5906	2.2157	0.2622
0.900	1.4775	2.2377	0.1841
0.950	1.3911	2.2483	0.1379

The Determination of ε

The quantity $\varepsilon (= k_\infty/\alpha)$ is functionally related to ρ. In the case of isotropic scattering the relation is well known and of a particularly simple structure (cf. Ref. [43]). In general, ε is determined by viewing it as an eigenvalue of the integral equation (20). There is an alternate way, however, to characterize ε which, while not the most analytically direct way, is perhaps of greatest value in generating an insight into the physical signficance of ε and also of supplying a link between ε and the directly observable quantities of the light in real media. This alternate characterization of ε stems from the following functional relation which holds between $K(z,\pm)$ and the various scattering and absorption function of an arbitrary medium ((31) of Sec. 9.2):

$$1 = \frac{b(z,-)}{K(z,-) - a(z,-)} - \frac{b(z,+)}{K(z,+) + a(z,+)} .$$

As depth is increased each term, as a result of the asymptotic property of the light field, tends toward a well-defined limit, so that as $z \to \infty$, the above relation tends to:

$$1 = \frac{b(-)}{k_\infty - D(-)a} - \frac{b(+)}{k_\infty + D(+)a} .$$

This may be rewritten as:

$$1 = \frac{\beta(-)}{\varepsilon - (1 - \rho)D(-)} - \frac{\beta(+)}{\varepsilon + (1 - \rho)D(+)} , \qquad (32)$$

which is the *general characteristic equation for* ε . Here we have written:

$$"\beta(\pm)" \quad \text{for} \quad \frac{\frac{1}{4\pi}\int_{\Xi_\mp}\int_{\Xi_\pm} g_o(\theta')p(\theta',\phi';\theta,\phi)\,d\Omega'\,d\Omega}{\pm\int_{\Xi_\pm} g_o(\theta')\cos\theta'\,d\Omega} \quad .$$

In the case of isotropic scattering:

$$\beta(\pm) = \frac{\rho}{2}\,D(\pm) \quad ,$$

and (32) reduces to the following simple form after the explicit expressions for $D(\pm)$, as given by (30), are substituted in it:

$$\rho = \frac{2\varepsilon}{\ln\left[\frac{1+\varepsilon}{1-\varepsilon}\right]} \quad . \tag{33}$$

This is the well-known characteristic equation for ε in the isotropic case. As ρ varies from 0 to 1, ε varies from 1 to 0. Hence, for all ρ , $0 \le \rho \le 1$; $0 \le \varepsilon \le 1$. Whenever scattering is present, i.e., whenever $\rho > 0$, then the useful inequality $k_\infty < \alpha$ holds. Actually, the inequalities $0 \le k_\infty/\alpha \le 1$ hold in general (Sec. 10.6). This fact is made plausible by an inspection of (21) keeping in mind that the function g_o is bounded in all physically meaningful situations, so that the denominator cannot vanish.

An Heurisitc Proof of the Hypothesis

We now present a brief argument which makes plausible the assertion of the hypothesis, namely that $K(z,\theta,\phi) \to k_\infty$ for all (θ,ϕ). For simplicity we will assume that the space is homogeneous and that scattering is isotropic. The resulting line of argument, while restricted to this special setting, can be made completely rigorous. The setting is that depicted in Fig. 10.15.

Under the present assumptions, we see that (18) may be written

$$N_q(z,\theta,\phi) = \frac{1}{4\pi}\,\rho h(z) \quad ,$$

so that:

$$K_q(z,\theta,\phi) = k(z) \quad .$$

Thus (19) reduces to

$$\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - k(z)][K(z,\theta,\phi) + \alpha \sec \theta] \quad .$$

The preceding discussion of this equation showed that $K(z,\theta,\phi)$ always tends toward $k(z)$ for downward directions. Hence if $k(z)$ approaches a limit, $K(z,\theta,\phi)$ also tends toward this limit. More explicitly, suppose there is some depth z_o below which $k(z)$ is essentially constant and equal to k_∞. Then the preceding equation is a simple Riccati equation for $K(z,\theta,\phi)$ whose general solution is:

$$K(z,\theta,\phi) = \frac{k_\infty + \alpha \sec \theta \; C \exp\{(k_\infty + \alpha \sec \theta)z\}}{1 - C \exp\{(k_\infty + \alpha \sec \theta)z\}} \tag{34}$$

where:

$$C = \frac{K(0,\theta,\phi) - k_\infty}{K(0,\theta,\phi) + \alpha \sec \theta} \quad .$$

Since:

$$k_\infty + \alpha \sec \theta < 0$$

for all $\theta > \pi/2$, it follows immediately from (34) that

$$\lim_{z\to\infty} K(z,\theta,\phi) = k_\infty$$

for all $\theta > \pi/2$. This means that the shape of the downwelling radiance distribution becomes fixed at great depths. It follows from the principles of invariance that the reflected upwelling radiance distribution also becomes fixed, so that the shape of the entire radiance distribution becomes fixed at great depths.

A Criterion for Asymptoticity

According to (34), $K(z,\theta,\phi)$ approaches k_∞ with *least* speed when $\theta = \pi$ (i.e., for the directly downward direction, as in Fig. 10.15). Hence when $K(z,\pi,\phi)$ has come within a given distance of k_∞, we can conclude that the other values $K(z,\theta,\phi)$, $\pi/2 \le \theta < \pi$ are within the same neighborhood of k_∞. From (34) it follows that

$$K(z,\pi,\phi) - k_\infty = \frac{(k_\infty - \alpha)C \exp (k_\infty - \alpha)z}{1 - C \exp (k_\infty - \alpha)z} \quad . \tag{35}$$

Thus a preassigned value of the difference on the left side determines an associated value of z. Although (35) is exact only at great depths, and applies only in the present (isotropic) context, it nevertheless supplies a useful approximate method for estimating the depths at which $K(z,\pi,\phi) - k_\infty$ has attained a given small value.

10.8 Simple Formulas for the Volume Absorption Coefficient in Asymptotic Light Fields

Introduction

As a result of the work of the preceding sections, it is now a well-established fact that the properties of the light field in optically deep homogeneous stratified media (such as deep cloud layers, oceans, lakes, etc.) become extremely regular and predictable at great depths. The conceptual and practical consequences of this fact were begun to be explored in Sec. 10.7; and the discussion there entered only in the first stages of exploration. In this section we derive some further consequences from this regularity property of deep light fields in the context of natural hydrosols. In particular, we use the regularity of the deep light field to derive several simple, exact, formulas relating the volume absorption coefficient to the common lighting value k_∞ of the K-functions for irradiance. In this way we supplement the exact formula

$$a(z) = \frac{1}{h(z)} \frac{d\bar{H}(z,+)}{dz} \tag{1}$$

for the volume absorption function α (as given in (18) of Sec. 8.8) with several alternate formulas which are especially suitable for engineering calculations and as handy rules of thumb relating a and k_∞. These formulas are as follows:

I. $$\boxed{a = k_\infty \frac{\bar{H}(z,-)}{h(z)} \quad , \quad z \geq z_o} \quad ,$$

II. $$\boxed{a = k_\infty \frac{(1 - R_\infty)}{D(-) + R_\infty D(+)}} \quad ,$$

III. $$\boxed{a = \frac{3}{4} k_\infty} \quad ,$$

where (as defined earlier in Sec. 10.7):

k_∞ is the common limit, as $z \to \infty$, of the K-functions $K(z,\pm)$, $k(z)$ for irradiance and scalar irradiance, respectively.

$D(\pm)$ are the limits, as $z \to \infty$, of the distribution functions $D(z,\pm)$.

R_∞ is the limit, as $z \to \infty$, of the reflectance function $R(z,-) = H(z,+)/H(z,-)$.

$\overline{H}(z,-)$ $= H(z,-) - H(z,+)$, the net downward irradiance at any depth $z \geq z_0$, where z_0 is the depth below which the light field has essentially attained its asymptotic structure.

The details of the derivation of I, II, and II, will now be given.

Short Derivation of I

The short derivation of I starts with (1), and the fact that there exists a depth z_0 below which the logarithmic derivatives of $H(z,-)$, $H(z,+)$, and $h(z)$ are constant and equal to a common value k (see (22), (24), and (35) of Sec. 10.7). Therefore:

$$\frac{d\overline{H}(z,+)}{dz} = \frac{dH(z,+)}{dz} - \frac{dH(z,-)}{dz}$$

$$= -\, k_\infty\, H(z,+) + k_\infty H(z,-) = k_\infty \overline{H}(z,-) \quad , \qquad (2)$$

for all $z \geq z_0$. Hence:

$$a = k_\infty \frac{\overline{H}(z,-)}{h(z)} \quad .$$

Long Derivation of I

The long derivation of I is essentially an exercise in the use of the integrated form of the divergence relation for the light field vector ((33) of Sec. 8.8)

$$\overline{P}(s,-) = a\, v\, U(M) \quad , \qquad (3)$$

where M is *any* regularly or irregularly shaped region of the optical medium, S is its boundary, and $\overline{P}(S,-)$ is the net inward flux across S into M. $U(M)$ is the radiant energy content of M, v is the speed of light in M, and a is the required value of the volume absorption coefficient.

It is interesting to observe that (3) yields a value of a in *any* homogeneous medium, regardless of the structure of the light field:

$$\boxed{a = \frac{\overline{P}(S,-)}{v\, U(M)}} \qquad (4)$$

The numerator of (4) can be obtained by traversing the boundary of M with flat plate collectors or other flux-measuring devices. The denominator is obtained by probing the interior of M with a spherical collector (to find $h(p)$ at each point p) and integrating the values over M.

In the present case, the extreme regularity of the asymptotic light field allows one to estimate $U(M)$ *knowing only one value* of the scalar irradiance at a boundary point of M. This fact holds also for $\overline{P}(S,-)$. Specifically, consider a region M in the form of a vertical column of unit cross section, and bounded by two parallel planes at depth z_1, and z_2, such that $z_0 \le z_1 \le z_2$. The medium is homogeneous and stratified; hence:

$$\overline{P}(s,-) = \overline{H}(z_1,-) + \overline{H}(z_2,+) \quad . \tag{5}$$

The net fluxes over the vertical sides of the column cancel by virtue of the stratified light field. By hypothesis, we have:

$$H(z_2,\pm) = H(z_1,\pm)e^{-k_\infty(z_2 - z_1)} \quad , \tag{6}$$

so that:

$$\overline{P}(s,-) = \overline{H}(z_1,-)\left[1 - e^{-k_\infty(z_2 - z_1)}\right] \quad . \tag{7}$$

Furthermore:

$$\begin{aligned} vU(M) &= \int_{z_1}^{z_2} h(z)dz \\ &= h(z_1)\int_{z_1}^{z_2} e^{-k_\infty(z - z_1)}\, dz \\ &= \frac{h(z_1)}{k_\infty}\left[1 - e^{-k_\infty(z_2 - z_1)}\right] \quad . \end{aligned} \tag{8}$$

Inserting (7) and (8) into the general formula (4), we have the desired result

$$a = k_\infty \frac{\overline{H}(z_1,-)}{h(z_1)} \quad , \quad z_1 \ge z_0 \quad .$$

Derivation of II

The formula II can be obtained directly from I by recalling that:

$$\bar{H}(z,-) = H(z,-) - H(z,+) \quad ,$$

$$h(z) = h(z,-) + h(z,+) \quad ,$$

and invoking the definitions of $R(z,-)$, and $D(z,-)$. That is, in general:

$$\bar{H}(z,-) = H(z,-) - R(z,-)H(z,-) = H(z,-)[1 - R(z,-)]$$

and

$$h(z) = D(z,-)H(z,-) + D(z,+)H(z,+) \quad ;$$

so that when $z \geq z_o$, we have:

$$a = k_\infty \frac{(1 - R_\infty)}{D(-) + R_\infty D(+)} \quad ,$$

which is the desired alternate formula. We observe in passing that II is a limiting form of the exact formula:

$$a(z) = \frac{K(z,-) - R(z,-)K(z,+)}{D(z,-) + R(z,-)D(z,+)} \quad , \tag{9}$$

the basis for which is (25) of 9.2. Clearly, as $z \to \infty$, equation (9) takes the limiting form II. Furthermore if we assume $D(\pm) = 2$, as is done in the classical one-D two-flow theory of the light field, then II reduces to the relation:

$$a = \frac{k_\infty}{2} \cdot \frac{1 - R_\infty}{1 + R_\infty} \quad . \tag{10}$$

Applied Numerology: A Rule of Thumb

Formula III is to be taken as a convenient rule of thumb, and as such, is subject to possible revision whenever specific optical media are under study. Yet for many purposes it is quite adequate, a fact which is based on the following observed regularities in the values of R_∞ and $D(\pm)$ in natural waters: R_∞ is usually found to be in the neighborhood of 0.02, give or take 0.01 for wavelengths near 500 μm . Furthermore for the same wavelength vicinity, $D(\pm)$ appears to be such that the sum $D(+) + D(-)$ is usually very nearly equal to 4; and the ratio $D(+)/D(-)$ is usually very nearly equal to 2, over great ranges of depths and in many media. Solving these two simultaneous equations yields, to two significant figures:

$$\begin{aligned} D(-) &= 4/3 \\ D(+) &= 8/3 \end{aligned} \tag{11}$$

which agrees very well with experimental results (cf., e.g., Table 1 of Sec. 8.5). It follows that, to the nearest rational

number with small integers for numerator and denominator, we have from II:

$$\boxed{a = \frac{3}{4} k_\infty} \tag{12}$$

or:

$$k_\infty = \frac{4}{3} a \quad . \tag{13}$$

Any similarity between the appearance of the fraction 4/3 in (13) and the index of refraction of water must be viewed as an amusing coincidence. Equation (13), incidentally, points up once again the kinship of k_∞ with the absorption mechanisms in optical media (see the discussion of (5) of Sec. 9.2 and (29) of Sec. 9.3).

10.9 Bibliographic Notes for Chapter 10

The developments of Secs. 10.1 to 10.4 are based on the work of [245].

The problem of the asymptotic light field in natural hydrosols was first clearly recognized by Whitney (re: [315] and [316]). The mathematical formulations and solutions of the problem as in Secs. 10.5, 10.6, and 10.7 are based on the researches in [224], [225], [244], and [226], respectively. Important references to the asymptotic radiance hypothesis in the hydrologic optics context may be found in [107], [108], and [209]. References to the asymptotic radiance hypothesis in the astrophysical context may be found in [43] and [147]; references to the neutron diffusion setting are made in [62]. Section 10.8 is based in the main on [230].

Experimental data in [298] exhibit clearly the asymptotic property of radiance fields in a real optical medium and were instrumental in the empirical establishment of the hypothesis.

CHAPTER 11

THE UNIVERSAL RADIATIVE TRANSPORT EQUATION

"All these examples, which might be multiplied by the millions, are cases in which a long, laborious, conscious, detailed process of acquirement has been condensed into...one. Factors which formerly had to be considered one by one in succession are integrated into what seems a single simple factor."

(*From:* "The Miracle of Condensed Recapitulation"
in the Preface of *Back to Methuselah*
Bernard Shaw)

11.0 Introduction

The present chapter concludes the development of the basic theory of radiative transfer in the present work with a survey of the manifold transport equations for the radiometric concepts introduced during Parts I, II, and the preceding chapters of Part III. The main purpose of the survey is to bring to light, especially for those readers interested in the theoretical aspects or radiative transfer, a recurrent symbolic theme which runs through every transport equation considered so far, and to go on to capture its essence in the form of a "universal radiative transport equation."

The universal radiative transport equation is an equation which, by suitable choice of its parameters, yields in turn such equations as the general equation of transfer for radiance, the general two-flow transport equations for irradiance, the transport equation for scalar irradiance, and the transport equations governing the apparent optical properties of an optical medium.

The primary purpose of the universal radiative transport equation is to formulate in a single mathematical package all the important transport equations which have evolved during the past seventy years in the theoretical studies of the steady state transfer of radiance energy through scattering-absorbing media of the stratified plane-parallel type. In this way a recapitulation of the evolutionary process of the transport equation's growth is achieved and a unification of all these important transport equations is attained. We shall illustrate the scope of the equation by selecting thirty-four

types of transport equations discussed in this work or implied by the discussions of their principal functions, and showing how these various types may be uniformly subsumed under the regime of the universal transport equation.

A second purpose of the universal transport equation is to provide a new useful tool in the study of radiative transfer theory. For example, certain special forms of the universal transport equation have already been successfully used (Secs. 10.5 and 10.7) to obtain a solution to the long-standing practical problem of the existence of the asymptotic light field in deep stratified hydrosols, a mathematical task which appears to be simplified, and given interesting physical significance with the introduction of the general type of functions associated with the universal transport equation. Further evidence of the usefulness of the universal transport equation as a tool which leads to new practical results will be illustrated below.

Before we go into the details of how the universal transport equation can achieve a semblance of unity in the classification of modern radiative transport equations, and of how it leads in some cases to new results which are beyond the immediate capabilities of the classical transport equations, it may be of help to the reader to indicate the steps in the development of modern radiative transfer theory which have led to the idea of the universal transport equation. With such information in mind the reader can then easily follow the steps of the synthesis.

There are four well-defined steps in the development of modern radiative transfer theory which form the immediate background to the formulation of the universal transport equation. These are, in chronological order: The adoption of the general equation of transfer for radiance and the development of the notion of equilibrium radiance [279], [111], and [43]; the development of the unified two-flow irradiance equations and the notion of equilibrium irradiance as recorded in Chapter 8; the development of the canonical equation of transfer and the notion of the radiance K-function as recorded in Chapter 4; the development of the theory of the asymptotic light field and the transport equation for the radiance K-function as recorded in Chapter 10.

In the following two sections we will illustrate these steps in detail and add still further illustrations which have been uncovered subsequent to the time of the fourth step. In this way we will systematically build up evidence for the existence of a universal transport equation and for the equilibrium principle (described below) with which it is closely associated. After these concrete examples of the various transport equations have been assembled, the genotype of the universal transport equation is extracted from them and displayed ((1) of Sec. 11.3). The chapter closes with a brief survey of less common but equally important examples of transport equations which are also subsumed by the universal transport equation.

11.1 Transport Equations for Radiometric Concepts

In this section we will present the transport equations for the following six radiometric quantities used in the study of plane-parallel media: radiance function $N(z,\theta,\phi)$, upwelling and downwelling irradiance functions $H(z,\pm)$, upwelling and downwelling scalar irradiance functions $h(z,\pm)$, and the scalar irradiance function $h(z)$.

Each of these transport equations is cast into a form which explicitly exhibits a certain attenuation function and equilibrium function associated with the radiometric concept it governs. It is the isolation and emphasis of these two concepts which is the earmark of the universal radiative transport equation. Thus, for example, the customary form of the equation of transfer for radiance is recast so that it explicitly exhibits the special attenuation function $-\alpha(z)/\cos\theta$ and the equilibrium function $N_q(z,\theta,\phi) = N_*(z,\theta,\phi)/\alpha(z)$. Similarly, the unified irradiance equations governing $H(z,\pm)$ are recast into forms which explicitly exhibit the corresponding attentuation functions $\mp [a(z,\pm) + b(z,\pm)]$ and equilibrium functions $H_q(z,\pm)$. These two reformulations for the transport equations of $N(z,\theta,\phi)$ and $H(z,\pm)$ are already known (see Sec. 10.7 for the case of N, and Sec. 8.3 for the case of H); however, the reformulations are now viewed with the purpose of seeing what mathematical and physical characteristics are held in common by these transport equations. It turns out that the common characteristics are the *attenuation* and *equilibrium* functions associated with each of the radiometric concepts governed by these equations and that each of these transport equations is but a special case of a more general equation, to be determined.

The discussion of the present section continues with the derivation of the exact transport equations for $h(z,\pm)$ and $h(z)$. It is shown that each of these functions also may have associated with it an *attenuation* function and an *equilibrium* function. In this way we show that the six radiometric quantities used in the study of plane-parallel media have an important set of properties common to all: The notion of an associated attenuation function and an associated equilibrium function, and finally that the transport equation for each of these six radiometric concepts, is subsumed under one general equation.

We now proceed to substantiate the preceding assertions by considering in turn each of the six radiometric concepts and its associated transport equation.

Equation of Transfer for Radiance

The equation of transfer for radiance ((3) of Sec. 3.15) in source-free stratified plane-parallel media is of the form:

$$-\cos\theta \frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)N(z,\theta,\phi) + N_*(z,\theta,\phi) \quad , \quad (1)$$

where:

$$N_*(z,\theta,\phi) = \int_{\Xi} N(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)\, d\Omega \quad .$$

Equation (1) is the most basic of all transport equations and, as we have seen repeatedly in the preceding chapters, can often be used in its full generality in the several different branches of applied radiative transfer theory such as astrophysical optics, and in the two subdisciplines of geophysical optics: hydrologic optics and meteorologic optics.

The reformulation of (1) which is of immediate interest is obtained by using the notion of equilibrium radiance:

$$N_q(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\alpha(z)} \quad , \tag{2}$$

for by means of this function, (1) may be written:

$$\boxed{\frac{dN(z,\theta,\phi)}{dz} = \frac{\alpha(z)}{\cos\theta}\,[N(z,\theta,\phi) - N_q(z,\theta,\phi)]} \tag{3}$$

Equation (3) is the desired reformulation of (1).* For our present purposes we draw special attention to the two functions:

$$\text{(i)} \quad -\frac{\alpha(z)}{\cos\theta}$$

$$\text{(ii)} \quad N_q(z,\theta,\phi) \tag{4}$$

Function (i) is the *attenuation function* for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ). Function (ii) is the *equilibrium function* for $N(z,\theta,\phi)$ for a fixed direction (θ,ϕ).

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*An alternate formulation of (3) is possible by adopting the optical depth parameter $\tau (= \int_0^z \alpha(z')dz')$. Such a formulation using τ has been found of especial use, e.g., in Chapter 10. However, for our present purposes, Eq. (3) is more appropriate.

Transport Equations for $H(z,\pm)$

The transport equations for $H(z,\pm)$ (or more accurately the *two-flow equations* for the irradiance field) are of the form (Chapter 8):

$$\mp \frac{dH(z,\pm)}{dz} = - [\alpha(z,\pm) + b(z,\pm)]H(z,\pm) + b(z,\mp)H(z,\mp) \tag{5}$$

Associated with $H(z,-)$ and $H(z,+)$ are the equilibrium functions $H_q(z,-)$ and $H_q(z,+)$, respectively. These equilibrium functions are defined by writing:

$$\text{"}H_q(z,\pm)\text{"} \quad \text{for} \quad \frac{b(z,\mp)H(z,\mp)}{a(z,\pm) + b(z,\pm)} \quad . \tag{6}$$

By means of these functions the equations in (5) may be written:

$$\boxed{\mp \frac{dH(z,\pm)}{dz} = - [a(z,\pm) + b(z,\pm)][H(z,\pm) - H_q(z,\pm)]} \ . \tag{7}$$

The equations in (7) are the desired reformulations of (5). For our present purposes we draw special attention to the two sets of functions:

$$\begin{aligned} &\text{(i)} \quad \mp [a(z,\pm) + b(z,\pm)] \\ &\text{(ii)} \quad H_q(z,\pm) \end{aligned} \tag{8}$$

Set (i) gives the *attenuation function* for the upwelling (+) and downwelling (-) irradiances $H(z,\pm)$. Observe that, by (11) of Sec. 8.3, the terms in (i) can be represented by $\pm\tau(z,\pm)$. Set (ii) gives the *equilibrium function* for the upwelling (+) and downwelling (-) irradiances $H(z,\pm)$.

Transport Equations for $h(z,\pm)$

The exact transport equations for $h(z,+)$ and $H(z,-)$ apparently have never been even remotely discussed in the literature. The reason for this gap in the family transport equations for the common radiometric concepts is two-fold. First, and perhaps most important, in the classical one-D theory, there has never been an explicit need for the transport equations for $h(z,\pm)$; the ordinary irradiances $H(z,\pm)$ were considered adequate in the early studies of the light field in stratified media. However, with the advent of more precise and detailed studies of the irradiance field (Chapters 8, 9, 10), the functions $h(z,\pm)$ have finally

assumed a legitimate and useful role in modern radiative transfer theory. Second, there is no simple or intuitively obvious way of obtaining the *exact* transport equations for $h(z,\pm)$ from first principles (that is, obtaining *de novo* derivations starting only with the definition of $h(z,\pm)$ and the basic volume absorption and volume scattering functions) as is the case for the irradiances $H(z,\pm)$. Neither is there any simple way of obtaining the requisite transport equations directly from the equation of transfer for radiance (again in contradistinction to case for $H(z,\pm)$). In the present paragraph we derive the exact transport equations for $h(z,\pm)$ by a simultaneous use of (a): The connections between these functions and $H(z,\pm)$, provided by the distribution functions $D(z,\pm)$; and (b): the exact transport equations for $H(z,\pm)$.

We begin with the derivation of the transport equation for $h(z,-)$. By definition of $D(z,-)$,

$$h(z,-) = D(z,-)H(z,-) \quad . \tag{9}$$

Taking the derivative of each side with respect to z :

$$\frac{dh(z,-)}{dz} = D(z,-)\,\frac{dH(z,-)}{dz} + H(z,-)\,\frac{dD(z,-)}{dz} \quad .$$

By means of (5), this may be written:

$$\frac{dh(z,-)}{dz} = D(z,-)\left\{[a(z,-) + b(z,-)]H(z,-) + b(z,+)H(z,+)\right\} + H(z,-)\,\frac{dD(z,-)}{dz} \quad .$$

Using the definitions of $D(z,-)$ and $D(z,+)$ $(= h(z,+)/H(z,+))$ and denoting the derivatives with respect to z by a prime (which will be used interchangeably with d/dz in all that follows), the preceding equation may be written:

$$h'(z,-) = \left\{-\,[a(z,-) + b(z,-)] + \frac{D'(z,-)}{D(z,-)}\right\} h(z,-) + \frac{D(z,-)}{D(z,+)}\,b(z,+)h(z,+) \quad , \tag{10}$$

which is the general transport equation for $h(z,-)$.

Now, as in the case of $N(z,\theta,\phi)$ and $H(z,\pm)$, we may associate with $h(z,-)$ an equilibrium function $h_q(z,-)$ where we write:

$$\text{"}h_q(z,-)\text{"} \quad \text{for} \quad \frac{\dfrac{D(z,-)}{D(z,+)}\,b(z,+)h(z,+)}{[a(z,-) + b(z,-)] - \dfrac{D'(z,-)}{D(z,-)}} \quad . \tag{11}$$

An alternate representation of $h_q(z,-)$ is:

$$h_q(z,-) = \frac{D^2(z,-)b(z,+)h(z,+)}{D(z,+)D(z,-)[a(z,-)+b(z,-)] - D'(z,-)D(z,+)} \quad .$$

With this definition of $h_q(z,-)$, the transport equation (10) may be written:

$$\boxed{\frac{dh(z,-)}{dz} = \left[-(a(z,-)+b(z,-)) + \frac{D'(z,-)}{D(z,-)}\right][h(z,-) - h_q(z,-)]} \tag{12}$$

Equation (12) is the reformulation of (11) which is of central interest in the present study, and as before we call special attention to the two functions:

$$\begin{aligned}&\text{(i)} \quad + \left[a(z,-)+b(z,-)\right] - \frac{D'(z,-)}{D(z,-)}\\ &\text{(ii)} \quad h_q(z,-)\end{aligned} \tag{13}$$

The function (i) is the *attenuation function* for $h(z,-)$. The function (ii) is the *equilibrium function* for $h(z,-)$.

The derivation of the transport equation for $h(z,+)$ proceeds in a similar manner to that leading to (12) and (13) in the case of $h(z,-)$. Therefore, the reader may easily verify first of all that:

$$-\frac{dh(z,+)}{dz} = \left\{[a(z,+)+b(z,+)] + \frac{D'(z,+)}{D(z,+)}\right\} h(z,+) + \frac{D(z,+)}{D(z,-)} b(z,-)h(z,-) \quad . \tag{14}$$

Next, if we write:

$$\text{"}h_q(z,+)\text{"} \quad \text{for} \quad \frac{\frac{D(z,+)}{D(z,-)} b(z,-)h(z,-)}{\left[a(z,+)+b(z,+)\right] + \frac{D'(z,+)}{D(z,+)}} \tag{15}$$

then we have also:

$$h_q(z,+) = \frac{D^2(z,+)b(z,-)h(z,-)}{D(z,+)D(z,-)[a(z,+)+b(z,+)] + D'(z,+)D(z,-)}$$

so that (14) may be written:

$$\boxed{-\frac{dh(z,+)}{dz} = -\left[(a(z,+)+b(z,+)) + \frac{D'(z,+)}{D(z,+)}\right]\left[h(z,+) - h_q(z,+)\right]} \quad , \tag{16}$$

which is the desired reformulation of (14). We draw special attention to the functions:

$$\text{(i)}\ -[a(z,+)+b(z,+)] - \frac{D'(z,+)}{D(z,+)}$$

$$\text{(ii)}\ h_q(z,+) \tag{17}$$

The function (i) is the *attenuation function* for $h(z,+)$. The function (ii) is the *equilibrium function* for $h(z,+)$.

We pause to observe the similarity of the functions in (8) (the set for $H(z,\pm)$) and with those in (13) and (17) (the set for $h(z,\pm)$). These sets coincide when $D'(z,\pm) = 0$, i.e., when $H(z,\pm)$ and $h(z,\pm)$ differ multiplicatively by a constant factor. That is, under this condition, (i) of (8) reduces to (i) of (13) and (17), and

$$\frac{H_q(z,\pm)}{h_q(z,\pm)} = D(z,\pm) = D(\pm), \quad \text{for all } z .$$

The physical significance of the condition $D'(z,\pm) = 0$ is now clear from the study of the two-D model for irradiance fields in Chapter 8, in particular from the introductory discussions of Sec. 8.5.

Transport Equation for Scalar Irradiance

To obtain the transport equation for the scalar irradiance function $h(z)$, we begin by decomposing $h(z)$ into its upwelling and downwelling components:

$$h(z) = h(z,+) + h(z,-) \quad .$$

Then by using the definitions of the distribution functions:

$$D(z) = \frac{h(z,\pm)}{H(z,\pm)} \quad , \quad .$$

$h(z)$ may be represented in terms of $D(z,\pm)$ and $H(z,\pm)$:

$$h(z) = D(z,-)H(z,-) + D(z,+)H(z,+) \quad .$$

Taking the derivative of $h(z)$, we have

$$\frac{dh(z)}{dz} = D(z,-)\,\frac{dH(z,-)}{dz} + H(z,-)\,\frac{dD(z,-)}{dz}$$

$$+ D(z,+)\,\frac{dH(z,+)}{dz} + H(z,+)\,\frac{dD(z,+)}{dz} \quad .$$

We now make use of the exact transport equations for $H(z,\pm)$:

$$\frac{dh(z)}{dz} = D(z,-)\left\{-\ [a(z,-) + b(z,-)]H(z,-) + b(z,+)H(z,+)\right\}$$

$$+ H(z,-)D'(z,-) + H(z,+)D'(z,+)$$

$$+ D(z,+)\left\{[a(z,+) + b(z,+)]H(z,+) - b(z,-)H(z,-)\right\} \quad .$$

The next step is to convert the products $D(z,\pm)H(z,\pm)$ into the equivalent functions $h(z,\pm)$ and write $h'(z)$ as a linear combination of $h(z,+)$, $h(z,-)$:

$$\frac{dh(z)}{dz} = -\ [a(z,-) + b(z,-)]h(z,-) + \frac{D(z,-)}{D(z,+)}\,b(z,+)h(z,+)$$

$$+ \frac{D'(z,-)}{D(z,-)}\,h(z,-) + \frac{D'(z,+)}{D(z,+)}$$

$$+ [a(z,+) + b(z,+)]h(z,+) - \frac{D(z,+)}{D(z,-)}\,b(z,-)h(z,-) \quad .$$

Collecting coefficients of $h(z,\pm)$:

$$\frac{dh(z)}{dz} = A_-(z)h(z,-) + A_+(z)h(z,+) \quad , \tag{18}$$

where we have written:

$$\text{"}A_-(z)\text{"} \quad \text{for} \quad -\ [a(z,-) + b(z,-)] + \frac{D'(z,-) - D(z,+)b(z,-)}{D(z,-)}$$

and:

$$\text{"}A_+(z)\text{"} \quad \text{for} \quad [a(z,+) + b(z,+)] + \frac{D'(z,+) + D(z,-)b(z,+)}{D(z,+)}$$

Evidently (18) is unchanged if we write:

$$\frac{dh(z)}{dz} = A_-(z)h(z,-) + A_-(z)h(z,+)$$

$$+ A_+(z)h(z,+) + A_+(z)h(z,-)$$

$$-\ [A_-(z)h(z,+) + A_+(z)h(z,-)] \quad .$$

But then this equation may be reduced to:

$$\frac{dh(z)}{dz} = [A_-(z) + A_+(z)]h(z) - [A_-(z)h(z,+) + A_+(z)h(z,-)] \tag{19}$$

which is the transport equation for $h(z)$.

By writing:

$$"h_q(z)" \quad \text{for} \quad \frac{A_-(z)h(z,+) + A_+(z)h(z,-)}{A_-(z) + A_+(z)} \quad ,$$

Equation (19) is expressible as:

$$\boxed{\frac{dh(z)}{dz} = [A_-(z) + A_+(z)]\ [h(z) - h_q(z)]} \quad . \tag{20}$$

For our present purposes, Equation (20) is of central interest, and we mark for future reference:

$$\begin{array}{ll} \text{(i)} & -\ [A_-(z) + A_+(z)] \\ \text{(ii)} & h_q(z) \end{array} \tag{21}$$

Expression (i) is the *attenuation function* for $h(z)$. Expression (ii) is the *equilibrium function* for $h(z)$.

Preliminary Unification and Preliminary Statement of the Equilibrium Principle

We have now reached a point in our discussion where we may consolidate the results obtained so far. The consolidation will serve two purposes: It will yield a preliminary view of the structure of the universal transport equation, and secondly, it will prepare the way for a discussion of the transport equations for the apparent optical properties to be taken up in the next section.

We turn now to the transport equations discussed so far, in particular the equations (3), (7), (12), (16), and (20). These six equations have a common mathematical structure, and the various components of the structure are associated with physical concepts common to the respective radiometric concepts. Specifically, let the general symbol "$\mathcal{P}(z)$" denote any one of the following six radiometric concepts:

$$\mathcal{P}(z): \quad \left\{ \begin{array}{l} N(z,\theta,\phi) \\ H(z,\pm) \\ h(z,\pm) \\ h(z) \end{array} \right.$$

Furthermore, let "$\mathcal{P}_\alpha(z)$" denote the associated attenuation function for $\mathcal{P}(z)$. Finally, let "$\mathcal{P}_q(z)$" denote the associated equilibrium function for $\mathcal{P}(z)$. Then each of the six transport equations developed above is precisely of the form:

$$\boxed{\frac{d\mathcal{P}(z)}{dz} = - \mathcal{P}_\alpha(z)\ [\mathcal{P}(z) - \mathcal{P}_q(z)]} \quad . \tag{22}$$

We now may make a key observation on the dynamic behavior of the five radiometric concepts which are associated with a general direction of flow (h(z) is the only one of the preceding concepts which, by definition, is not associated with any particular directed pencil of radiation or general hemispherical flow). If "$\mathcal{P}(z)$" stands for any one of these five concepts: $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, then it is easy to verify that on the basis of (22):

$$\text{If } \mathcal{P}(z) > \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d|z|} < 0 \quad ,$$

and: (23)

$$\text{if } \mathcal{P}(z) < \mathcal{P}_q(z), \text{ then } \frac{d\mathcal{P}(z)}{d|z|} > 0 \quad ,$$

where the symbol '$d\mathcal{P}(z)/d|z|$' is defined as follows, we write:

$$\text{"}\frac{d\mathcal{P}(z)}{d|z|}\text{" for } \frac{d\mathcal{P}(z)}{dz}$$ if $\mathcal{P}(z)$ is associated with the direction of increasing z (downwelling direction)

and:

$$\text{"}\frac{d\mathcal{P}(z)}{d|z|}\text{" for } \frac{d\mathcal{P}(z)}{d(-z)}$$ if $\mathcal{P}(z)$ is associated with the direction of decreasing z (upwelling direction).

In other words, the equations (23) simply state that as the geometric form of the radiation represented by $\mathcal{P}(z)$ travels *in its assigned direction*, the magnitude of $\mathcal{P}(z)$ always changes in such a way that it tends to approach the magnitude of its equilibrium function $\mathcal{P}_q(z)$. This observation forms the core of the general equilibrium principle formulated below.

11.2 Transport Equations for Apparent Optical Properties

The notion of "apparent optical property" is discussed in detail in Chapter 9. The following list consists of the ten more important apparent optical properties associated with plane-parallel media, as developed in Chapter 9:

$$\begin{cases} K(z,\xi) \\ K(z,\pm) \\ R(z,\pm) \\ D(z,\pm) \\ k(z,\pm) \\ k(z) \end{cases}$$

We shall show in this section that a transport equation may be assigned to each of the above K - functions. As noted, we can assign a transport equation to either of the reflectance functions $R(z,\pm)$ and to the distribution functions $D(z,\pm)$, and in fact we will exhibit the transport equation for $R(z,-)$ and go on to deduce, by means of this equation, an interesting property about the depth behavior of $R(z,-)$. We will not, however, exhibit the transport equation for $R(z,+)$ and $D(z,\pm)$ for the following reasons: By definition, $R(z,+) = 1/R(z,-)$, so that once a transport equation is obtained for $R(z,-)$, one for $R(z,+)$ would be superfluous. The reason for not obtaining transport equations for the optical properties $D(z,\pm)$ is more subtle and may be inferred from the preceding formulations by recalling that the transport equations for $H(z,\pm)$, $h(z,\pm)$ make implicit or explicit use of the distribution functions. If we were to deduce the transport equations for $D(z,\pm)$ we would see that the quantities $H(z,\pm)$ or $h(z,\pm)$ would be explicitly involved in them. Therefore, a logical circularity would creep into the final set of transport equations if we insisted on obtaining transport equations for $D(z,\pm)$ in addition to those of $H(z,\pm)$ and $h(z,\pm)$. In order to avoid such a circularity we must decide on the elimination of one of the three sets of quantities: $H(z,\pm)$, $h(z,\pm)$, $D(z,\pm)$. Such a decision is easy to reach after we note that $H(z,\pm)$ and $H(z,\pm)$ are the fundamental observables in natural light fields, and that the $D(z,\pm)$ simply act as analytical liaisons between these quantities. Therefore, we will agree that $D(z,\pm)$ are to continue to act as the connecting links between the irradiance and scalar irradiance concepts, and that they are to enter into the calculations solely in the capacity of dimensionless mathematical parameters. Their usual physical interpretation will, of course, be retained, namely that they are measures of the directional variation of the radiance distribution at a general depth z. (In this connection, see Sec. 8.5.)

Canonical Forms of Transport Equations for K Functions

The procedure for obtaining the the transport equation for the six K-functions is facilitated by the preceding results, in particular by means of the six transport equations for $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, and $h(z)$. If "$\mathscr{P}(z)$" denotes for any of these six functions, then the corresponding K - function $K(\mathscr{P})$ is defined by writing:

$$\text{"}K(\mathcal{P})\text{"} \quad \text{for} \quad -\frac{1}{\mathcal{P}(z)}\frac{d\mathcal{P}(z)}{dz} \quad . \tag{1}$$

Using the generic equation (22) of Sec. 11.1 and the definition (1), we have:

$$-\mathcal{P}(z)K(\mathcal{P}) = -\mathcal{P}_\alpha[\mathcal{P}(z) - \mathcal{P}_q(z)] \quad .$$

Solving this for $\mathcal{P}(z)$, we obtain the *canonical form* of the transport equation for $\mathcal{P}(z)$:

$$\boxed{\mathcal{P}(z) = \frac{\mathcal{P}_q(z)}{1 - \left[\dfrac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right]}} \quad , \tag{2}$$

The canonical form for the radiance function of Chapter 4 is thus extended to wider contexts (see, e.g., (5) of Sec. 4.7).

This canonical form of the transport equation serves as the common starting point for the derivation of the equations governing individual K-functions. Thus, by taking the formal logarithmic derivative of each side of (2):

$$\frac{d \ln \mathcal{P}(z)}{dz} = \frac{d \ln \mathcal{P}_q(z)}{dz} - \frac{d}{dz} \ln \left[1 - \frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right] \quad ,$$

and writing, in analogy to (1):

$$\text{"}K_q(\mathcal{P})\text{"} \quad \text{for} \quad -\frac{1}{\mathcal{P}_q(z)}\frac{d\,\mathcal{P}_q(z)}{dz} \quad , \tag{3}$$

we have:

$$-K(\mathcal{P}) = -K_q(\mathcal{P}) + \frac{\dfrac{d}{dz}\left[\dfrac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right]}{1 - \dfrac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}} \quad ,$$

whence:

$$\frac{d}{dz}\left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)}\right] = \left[\frac{K(\mathcal{P})}{\mathcal{P}_\alpha(z)} - 1\right]\left[K(\mathcal{P}) - K_q(\mathcal{P})\right] \quad . \tag{4}$$

As it stands, (4) may be taken as the transport equation for $K(\mathcal{P})$. However, by suitable transformations of variables, we can reduce (4) to the general form of the universal transport equation. We next consider such transformations.

Dimensionless Transport Equation for $K(\mathcal{P})$

At the present point of the discussions (namely (4)), we have two alternative routes open to a universal transport equation: One route starts with the adoption of a generalized notion of *optical depth* defined by writing:

$$"\tau(z)" \quad \text{or} \quad "\tau" \quad \text{for} \quad \int_0^z \mathcal{P}_\alpha(z')dz' \quad ,$$

along with a *relativization* of $K(\mathcal{P})$ and $K_q(\mathcal{P})$ with respect to $\mathcal{P}_\alpha(z)$; thus we write:

$$"\tilde{K}(\mathcal{P})" \quad \text{for} \quad \frac{K(\mathcal{P})}{\mathcal{P}_\alpha} \quad ,$$

and:

$$"\tilde{K}_q(\mathcal{P})" \quad \text{for} \quad \frac{K_q(\mathcal{P})}{\mathcal{P}_\alpha} \quad .$$

Then (4) may be written in the *dimensionless form:*

$$\boxed{\frac{d\tilde{K}(\mathcal{P})}{d\tau} = [\tilde{K}(\mathcal{P}) - 1][\tilde{K}(\mathcal{P}) - \tilde{K}_q(\mathcal{P})]} \tag{5}$$

Equation (5) has the advantage of simplicity of structure and is therefore ideal for formal work. For example, the dimensionless form of (5) for the case $\mathcal{P}(z) = N(z,\theta,\phi)$ was used in the proof of the asymptotic radiance hypothesis in Sec. 10.5. However, (5) has the disadvantage of not showing the explicit effects on the associated K-functions produced by inhomogeneities of the medium nor of the way in which the K-functions vary with *geometrical depth,* the natural measure of depth used in experimental work. Therefore, we will actually take the second route which consists in adopting *geometrical depth* and which uses *un*relativized K-functions. This results in a mathematically more cumbersome transport equation but is actually of greater use in practical applications. By adopting the alternative route, we are now obliged to consider each of the K-functions in turn. The common starting point is (2) in which the explicit forms of $\mathcal{P}_\alpha(z)$ and $\mathcal{P}_q(z)$ for the various concepts have been substituted.

Transport Equation for $K(z,\theta,\phi)$

From (2) we have

$$N(z,\theta,\phi) = \frac{N_q(z,\theta,\phi)}{1 + \cos\theta \dfrac{K(z,\theta,\phi)}{\alpha(z)}} \quad , \tag{6}$$

in which we have set $\mathcal{P}_\alpha(z) = -\alpha(z)/\cos\theta$, $\mathcal{P}_q(z) = N_q(z,\theta,\phi)$, so that $K(\mathcal{P}) = K(z,\theta,\phi)$ and $K_q(\mathcal{P}) = K_q(z,\theta,\phi)$, using the definitions in (4) of Sec. 11.1. Taking the logarithmic derivative of each side of (6) and solving for $dK(z,\theta,\phi)/dz$:

$$\frac{dK(z,\theta,\phi)}{dz} = K^2(z,\theta,\phi) + \left[\frac{\alpha(z)}{\cos\theta} - K_q(z,\theta,\phi) + \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz}\right]K(z,\theta,\phi) - K_q(z,\theta,\phi)\frac{\alpha(z)}{\cos\theta} .$$

The right-hand side of this equation may be factored into the product of two functions yielding the desired form of the transport equation for $K(z,\theta,\phi)$:

$$\boxed{\frac{dK(z,\theta,\phi)}{dz} = [K(z,\theta,\phi) - \mathcal{K}_\alpha(z,\theta,\phi)][K(z,\theta,\phi) - \mathcal{K}_q(z,\theta,\phi)]} \ , \tag{7}$$

where $\mathcal{K}_\alpha$ and $\mathcal{K}_q$ are defined in context by the following two equations:

$$\begin{aligned}\mathcal{K}_\alpha(z,\theta,\phi) + \mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta} + K_q(z,\theta,\phi) - \frac{1}{\alpha(z)}\frac{d\alpha(z)}{dz}\\ \mathcal{K}_\alpha(z,\theta,\phi)\mathcal{K}_q(z,\theta,\phi) &= -\frac{\alpha(z)}{\cos\theta}K_q(z,\theta,\phi)\end{aligned} \tag{8}$$

The functions $\mathcal{K}_\alpha(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$ appearing in (7) are, respectively the attenuation and equilibrium functions for $K(z,\theta,\phi)$. They are defined as shown by the pair of simultaneous equations in (8), whose solutions are:

$$\left.\begin{matrix}2\mathcal{K}_q\\ \\ 2\mathcal{K}_\alpha\end{matrix}\right\} = -\left[\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right] \pm \left[\left(\frac{\alpha}{\cos\theta} - K_q + (\ln\alpha)'\right)^2 + \frac{4K_q\alpha}{\cos\theta}\right]^{1/2}$$

The quantities K_q and $\mathcal{K}_q$ should not be confused with each other. K_q is the logarithmic derivative of N_q (see definition (3)) whereas $\mathcal{K}_q$ is the sought-for equilibrium function for $K(z,\theta,\phi)$ in the general context. Observe, however, that if the medium were homogeneous, then:

$$\mathcal{K}_\alpha(z,\theta,\phi) = -\frac{\alpha}{\cos\theta} \ ,$$

$$\mathcal{K}_q(z,\theta,\phi) = K_q(z,\theta,\phi) \ .$$

More generally, in eventually homogeneous media (i.e., media in which $\alpha'(z) \to 0$ as $z \to \infty$)

$$\mathcal{K}_\alpha(z,\theta,\phi) \to -\frac{\alpha}{\cos\theta} \quad ,$$

$$\mathcal{K}_q(z,\theta,\phi) \to K_q(z,\theta,\phi) \to k_\infty \quad .$$

This follows from the asymptotic radiance theorem and its various consequences discussed in Chapter 10.

Transport Equations for K(z,±)

The appropriate form of (2) in the case of the K-functions $K(z,\pm)$ is obtained by substituting the attenuation and equilibrium functions for $H(z,\pm)$ in (2):

$$H(z,\pm) = \frac{H_q(z,\pm)}{1 \pm \dfrac{K(z,\pm)}{[a(z,\pm) + b(z,\pm)]}} \quad .$$

Taking the logarithmic derivatives of each side, solving for $K'(z,\pm)$, and factoring the quadratic in $K(z,\pm)$, we have

$$\boxed{\begin{aligned} \frac{dK(z,-)}{dz} &= [K(z,-) - \mathcal{K}_\alpha(z,-)][K(z,-) - \mathcal{K}_q(z,-)] \\ \frac{dK(z,+)}{dz} &= [K(z,+) - \mathcal{K}_\alpha(z,+)][K(z,+) - \mathcal{K}_q(z,+)] \end{aligned}} \quad \begin{aligned} &, \quad (9) \\ &. \quad (10) \end{aligned}$$

For $K(z,+)$ the functions $\mathcal{K}_\alpha(z,+)$, $\mathcal{K}_q(z,+)$ are defined in context by the equations:

$$\mathcal{K}_\alpha(z,+) + \mathcal{K}_q(z,+) = [a(z,-) + b(z,-)] + K_q(z,-) - (\ln[a(z,-) + b(z,-)])'$$

$$\mathcal{K}_\alpha(z,+)\mathcal{K}_q(z,+) = [a(z,-) + b(z,-)]K_q(z,-)$$

Similarly, for $K(z,-)$:

$$\mathcal{K}_\alpha(z,-) + \mathcal{K}_q(z,-) = -[a(z,+) + b(z,+)] + K_q(z,+) - (\ln[a(z,+) + b(z,+)])'$$

$$\mathcal{K}_\alpha(z,-)\mathcal{K}_q(z,-) = -[a(z,+) + b(z,+)]K_q(z,+) \quad .$$

These simultaneous equations may be solved to obtain explicit expressions for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_q$'s. We will not do this here, but simply point out that, in all eventually homogeneous media, as $z \to \infty$,

$$\mathcal{K}_\alpha(z,\pm) \to \mp[a(z,\pm) + b(z,\pm)] \quad ,$$

$$\mathcal{K}_q(z,\pm) \to K_q(z,\pm) \to k_\infty \quad .$$

This follows from the asymptotic radiance theorem and its various consequences studied in Chapter 10. As in the case of $K_q(z,\theta,\phi)$ and $\mathcal{K}_q(z,\theta,\phi)$, care should be taken so as not to confuse $K_q(z,\pm)$ with $\mathcal{K}_q(z,\pm)$. The former is defined in (3), the latter by the preceding simultaneous equations.

Transport Equations for $k(z,\pm)$ and $k(z)$

Starting with the general canonical equation (2), we have for $h(z)$:

$$h(z) = \frac{h_q(z)}{1 + \dfrac{k(z)}{[A_+(z) + A_-(z)]}} \quad .$$

Similarly, for $h(z,\pm)$:

$$h(z,\pm) = \frac{h_q(z,\pm)}{1 \pm \dfrac{k(z,\pm)}{[a(z,\pm) + b(z,\pm)] - \dfrac{D'(z,\pm)}{D(z,\pm)}}} \quad .$$

The existence of these canonical equations for $h(z,\pm)$ and $h(z)$ is sufficient to prove the existence of the appropriate transport equations for $k(z,\pm)$ and $k(z)$ by following the procedure illustrated in the preceding two paragraphs. The results are

$$\boxed{\begin{aligned} \frac{dk(z,\pm)}{dz} &= [k(z,\pm) - k_\alpha(z,\pm)][k(z,\pm) - k_q(z,\pm)] \\ \frac{dk(z)}{dz} &= [k(z) - k_\alpha(z)][k(z) - k_q(z)] \end{aligned}} \quad \begin{aligned} &, (11) \\ &. (12) \end{aligned}$$

The exact forms for the respective $\mathcal{K}_\alpha$'s and $\mathcal{K}_q$'s will not be worked out; this may be left as an exercise for the interested reader. The important point to observe is that we have now proved that for all six K-functions, the generic transport equation is:

$$\boxed{\frac{dK(\mathcal{P})}{dz} = [K(\mathcal{P}) - \mathcal{K}_\alpha(\mathcal{P})][K(\mathcal{P}) - \mathcal{K}_q(\mathcal{P})]} \qquad (13)$$

Equations (22) of Sec. 11.1 and (13) form the two major sets of transport equations considered in this chapter. These

two equations cover all twelve transport equations for $\mathcal{P}$ and $K(\mathcal{P})$ considered so far.

As in the case of (22) of Sec. 11.1, it is easy to verify on the basis of (13) that:

$$\text{If} \quad K(\mathcal{P}) > \mathcal{K}_q(\mathcal{P}), \quad \text{then} \quad \frac{dK(\mathcal{P})}{d|z|} < 0 \quad ,$$

and (14)

$$\text{if} \quad K(\mathcal{P}) < \mathcal{K}_q(\mathcal{P}), \quad \text{then} \quad \frac{dK(\mathcal{P})}{d|z|} > 0 \quad ,$$

which show that $K(\mathcal{P})$ always tends toward* its equilibrium function $\mathcal{K}_q(\mathcal{P})$.

We now turn to consider the last of the standard transport equations, namely that for $R(z,-)$.

Transport Equation for R(z,-)

By definition of $R(z,-)$:

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \quad .$$

Taking the logarithmic derivative of each side, and applying the definitions of $K(z,+)$ and $K(z,-)$, we have:

$$\frac{dR(z,-)}{dz} = R(z,-)[K(z,-) - K(z,+)] \quad .$$

Using the following representations (18) and (19) of Sec. 9.2 of $K(z,\pm)$:

$$K(z,\pm) = \mp [a(z,\pm) + b(z,\pm)] \pm b(z,\mp)R(z,\pm) \quad ,$$

the derivative of $R(z,-)$ may be cast into the form:

$$\frac{dR(z,-)}{dz} = - b(z,+)R^2(z,-) + [a(z,-) + a(z,+) + b(z,-) + b(z,+)]\cdot$$

$$\cdot R(z,-) - b(z,-) \quad .$$

The right-hand side, which is a quadratic in $R(z,-)$, may be factored:

$$\frac{dR(z,-)}{dz} = - b(z,+)[R(z,-) - R_\alpha(z,-)][R(z,-) - R_q(z,-)]. \quad (15)$$

---

*The term "tends toward" has a precise meaning here: If f_1 and f_2 are two real-valued functions defined on some common domain $\mathcal{D}$ of the reals then f_1 *tends toward* f_2 at $x \in \mathcal{D}$ if sign $[f_2(x) - f_1(x)]$ = sign $f_1'(x)$ where "sign" means the same as "sign of." As an earlier example of this, see (4) of Sec. 9.4.

Equation (15) is the required transport equation for $R(z,-)$, in which $R_\alpha(z,-)$ is the *attenuation function* for $R(z,-)$ and $R_q(z,-)$ is the *equilibrium function* for $R(z,-)$ (compare with (2) of Sec. 9.4). These functions are defined in context by the following system of simultaneous equations:

$$R_\alpha(z,-) + R_q(z,-) = \frac{a(z,-) + a(z,+) + b(z,-) + b(z,+)}{b(z,+)}$$

$$R_\alpha(z,-)R_q(z,-) = \frac{b(z,-)}{b(z,+)} \quad . \tag{16}$$

As in the case of the K - functions, these may be solved for $R_\alpha(z,-)$ and $R_q(z,-)$:

$$\left.\begin{matrix} 2R_\alpha(z,-) \\ \\ 2R_q(z,-) \end{matrix}\right\} = \left\{R(z,-) + \frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} - \frac{1}{b(z,+)}\,[K(z,-) - K(z,+)]\right\}$$

$$\pm \left[\{\;\}^2 - 4\,\frac{b(z,-)}{b(z,+)}\right]^{1/2} \quad . \tag{17}$$

R_α goes with the plus sign, R_q with the minus sign.

We observe that, in eventually homogeneous media, as $z \to \infty$:

$$R_\alpha(z,-) \to \frac{1}{R(z,-)}\,\frac{b(z,-)}{b(z,+)} \to \frac{1}{R_\infty}\,\frac{b(-)}{b(+)}$$

$$R_q(z,-) \quad \to \quad R(z,-) \quad \to \quad R_\infty \quad . \tag{18}$$

These facts follow from (17) and the asymptotic radiance theorem of Sec. 10.7.

11.3 Universal Radiative Transport Equation and the Equilibrium Principle

For the purposes of this section, let us refer to the thirteen quantities studied so far as the *standard concepts* (namely $N(z,\theta,\phi)$, $H(z,\pm)$, $h(z,\pm)$, $h(z)$, $K(z,\theta,\phi)$, $K(z,\pm)$, $k(z,\pm)$, $k(z)$, and $R(z,-)$). A *directed standard concept* is any of the preceding standard concepts except $h(z)$ and $K(z)$.

The evidence gathered in the preceding discussions may now be assembled in the form of:

THE UNIVERSAL RADIATIVE TRANSPORT EQUATION AND THE EQUILIBRIUM PRINCIPLE. *Let X be an arbitrarily stratified source-free plane-parallel medium with arbitrary incident lighting conditions. Let "$\mathcal{C}(z)$" denote any one of the standard concepts. Then associated with $\mathcal{C}(z)$ are two functions $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_q(z)$, the attenuation and equilibrium functions for $\mathcal{C}(z)$, respectively. The standard concept $\mathcal{C}(z)$ together with $\mathcal{C}_\alpha(z)$ and $\mathcal{C}_q(z)$ satisfy the functional relation:*

$$\frac{d\mathcal{C}(z)}{dz} = \mu(z)\left[\delta\mathcal{C}(z) - \mathcal{C}_\alpha(z)\right]\left[\mathcal{C}(z) - \mathcal{C}_q(z)\right] \quad , \qquad (1)$$

*where $\mu(z)$ and δ are known parameters depending on $\mathcal{C}(z)$. The relation (1) is the **universal radiative transport equation.** (*It is *degenerate* if $\delta = 0$; and *normalized* if $\mu = 1$, $\delta = 1$.)

If $\mathcal{C}(z)$ is a directed standard concept and $\mu(z) > 0$, *then:*

$$\frac{d\mathcal{C}(z)}{d|z|} \gtrless 0 \quad \text{whenever} \quad \mathcal{C}(z) \lessgtr \mathcal{C}_q(z) \quad ; \qquad (2)$$

and if $\mathcal{C}(z)$ is any standard concept, and X is eventually homogeneous, then:

$$\mathcal{C}_\alpha(\infty) = \lim_{z\to\infty}\mathcal{C}_\alpha(z) \qquad \textit{exists,}$$

$$(3)$$

$$\mathcal{C}_q(\infty) = \lim_{z\to\infty}\mathcal{C}_q(z) \qquad \textit{exists,}$$

and:

$$\lim_{z\to\infty}\mathcal{C}(z) = \mathcal{C}_q(\infty) \quad . \qquad (4)$$

The proof of the statements (1), (2), (3), and (4) have essentially been covered in the preceding discussions either directly (as in the case of (1)), or indirectly by references to the appropriate portions of the present work (as in the case of (2)-(4)). Table 1 below gives the explicit forms of $\mu(z)$ and δ for the thirteen standard concepts: An examination of Table 1 shows that if $R(z,-)$ is removed from the list of standard concepts, a considerable simplification is effected in the form of (1). However, in the interests of completeness we have included $R(z,-)$ within the purview of (1), and we note that by a change of z - scale, the equation is normalizable.

TABLE 1

Standard Cases of the Universal Radiative Transport Equation

Standard Concept	Values of μ,δ
$N(z,\theta,\phi)$ $H(z,\pm)$ $h(z,\pm)$ $h(z)$	$\mu(z) = 1$ $\delta = 0$ (degenerate)
$K(z,\theta,\phi)$ $K(z,\pm)$ $k(z,\pm)$ $k(z)$	$\mu(z) = 1$ $\delta = 1$ (normalized)
$R(z,-)$	$\mu(z) = -b(z,+)$ $\delta = 1$ (normalizable)

11.4 Some Additional Transport Equations Subsumed by the Universal Transport Equation

The standard transport equations enumerated in Table 1 of Sec. 11.3 constitute the most frequently used equations in general radiative transfer theory. This list, however, by no means exhausts the various ramifications of the universal transport equation as given by (1) of Sec. 11.3. An additional set of transport equations which fall under the domain of the degenerate universal transport equation will now be mentioned. This set is associated with less frequently used--but no less important--radiometric concepts than those of the standard type. We will consider in particular the following radiometric quantities:

(i)	n-ary radiance	N^n
(ii)	n-ary radiant energy	U^n
(iii)	path function	N_*
(iv)	vector irradiance	$\mathbf{H}$

(i) *The transport equation governing* N^n in plane-parallel media is ((1) of Sec. 5.2):

$$- \cos\theta \frac{dN^n(z,\theta,\phi)}{dz} = -\alpha(z)N^n(z,\theta,\phi) + N_*^n(z,\theta,\phi) \qquad (1)$$

where (re: (6) of Sec. 5.1):

$$N_*^n(z,\theta,\phi) = \int_\Xi N^{n-1}(z,\theta',\phi')\sigma(z;\theta',\phi';\theta,\phi)\,d\Omega \quad . \tag{2}$$

Here N^n, $n = 1, 2, \ldots,$ is the n-ary scattered radiance (re: (11) of Sec. 5.1), i.e., radiance consisting of photons having been scattered precisely n-times with respect to those comprising the initial radiance N^o entering the medium. In any particular problem, it is assumed that $N^o(z,\theta,\phi)$ is given. From this, $N_*^1(z,\theta,\phi)$ is obtainable by means of (2). Then $N_*^1(z,\theta,\phi)$ is known, and (51) becomes a differential equation in $N^1(z,\theta,\phi)$, which is easily solved in principle. This solution, we recall, is the basis of the definition (4) of Sec. 5.1. Numerical solutions of $N^1(z,\theta,\phi)$ may be readily obtained by means of a computer programmed for (1) (cf. concluding remarks in Sec. 5.6). Once $N^1(z,\theta,\phi)$ is known for all z and (θ,ϕ), (2) yields $N_*^2(z,\theta,\phi)$ and (1) may be solved for $N^2(z,\theta,\phi)$. By repeating this process, we are led to obtain $N^n(z,\theta,\phi)$ knowing $N^{n-1}(z,\theta,\phi)$. The total (observable) radiance $N(z,\theta,\phi)$ is defined by writing ((3) of Sec. 5.2):

$$\text{"}N(z,\theta,\phi)\text{"} \quad \text{for} \quad \sum_{n=o}^{\infty} N^n(z,\theta,\phi) \quad .$$

For our present purposes we write:

$$\text{"}N_q^n(z,\theta,\phi)\text{"} \quad \text{for} \quad \frac{N_*^n(z,\theta,\phi)}{\alpha(z)} \quad ,$$

so that (1) may be written:

$$\boxed{\frac{dN^n(z,\theta,\phi)}{dz} = \frac{\alpha(z)}{\cos\theta}\,[N^n(z,\theta,\phi) - N_q^n(z,\theta,\phi)]} \quad . \tag{3}$$

When written in this form, (3) closely parallels the form of (3) of Sec. 11.1, so that we conclude, as in (4) of Sec. 11.1:

(a) $-\dfrac{\alpha(z)}{\cos\theta}$ is the attenuation function for $N^n(z,\theta,\phi)$

(b) $N_q^n(z,\theta,\phi)$ is the equilibrium function for $N^n(z,\theta,\phi)$.

In this way the transport equation for $N^n(z,\theta,\phi)$ is subsumed by (1) of Sec. 11.3, in which $\mu(z) = 1$, $\delta = 0$.

(ii) *The time-dependent transport equation governing* U^n in a medium with no net flux across its boundary is usually written in terms of a time parameter t instead of a space parameter z ((24) of Sec. 5.8):

$$\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s} \tag{4}$$

where $T_\alpha = 1/v\alpha$, $T_s = 1/vs$. However, if "v" denotes the speed of light in X, then we may introduce a new variable r by writing

$$\text{"}r\text{"} \quad \text{for} \quad vt \quad ,$$

so that (4) becomes:

$$\frac{dU^n(r)}{dr} = -\alpha U^n(r) + sU^{n-1}(r) \quad . \tag{5}$$

The symbol "$U^n(r)$" denotes the n-ary radiant energy content of a sphere of radius r about a point source (in a space X) which emits radiant flux in some prescribed manner starting from time $t = 0$. The space is assumed homogeneous so that $\alpha(z) = \alpha$ for every z in the space. By writing:

$$\text{"}U^n_*(r)\text{"} \quad \text{for} \quad sU^{n-1}(r)$$

and:

$$\text{"}U^n_q(r)\text{"} \quad \text{for} \quad \frac{U^n_*(r)}{\alpha} \left(= \rho U^{n-1} \right)$$

where "ρ" denotes s/α, (5) may be then written:

$$\boxed{\frac{dU^n(r)}{dr} = -\alpha \left[U^n(r) - U^n_q(r) \right]} \tag{6}$$

Hence:

(a) α is the *attenuation function* for U^n

(b) U^n_q is the *equilibrium function* for U^n,

and (6) is subsumed by (1) of Sec. 11.3.

(iii) *The transport equation governing* N_* has the form (re: (9) of Sec. 5.2):

$$-\cos\theta \, \frac{dN_*(z,\theta,\phi)}{dz} = -\alpha N_*(z,\theta,\phi) + N_{**}(z,\theta,\phi) \tag{7}$$

where

$$N_{**}(z,\theta,\phi) = \int_\Xi N_*(z,\theta',\phi')\sigma(\theta',\phi';\theta,\phi)\,d\Omega \quad . \tag{8}$$

Equation (7) holds in all homogeneous isotropic media (thus the reason for explicity dropping z - notation in α and σ; on the other hand, σ may be arbitrary). If we write:

$$\text{"} N_{*q}(z,\theta,\phi) \text{"} \quad \text{for} \quad \frac{N_{**}(z,\theta,\phi)}{\alpha}$$

then:

$$\frac{dN_*(z,\theta,\phi)}{dz} = \frac{\alpha}{\cos\theta}\left[N_*(z,\theta,\phi) - N_{*q}(z,\theta,\phi)\right] \quad ; \tag{9}$$

therefore

(a) $-\dfrac{\alpha}{\cos\theta}$ is the *attenuation function* for N_* ,

(b) N_{*q} is the *equilibrium function* for N_* .

(iv) *The steady-state, source-free transport equation for vector irradiance* **H** has the form (the case $n = 2$ in (55) of Sec. 8.6):

$$\frac{d}{dz}\bar{H}(z,\mathbf{n},\Xi_o) = -\left[a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)\right]\bar{H}(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o')\bar{H}(z,\mathbf{n},\Xi_o') \tag{10}$$

Here we write:

$$\text{"}\bar{H}(z,\mathbf{n},\Xi_o)\text{"} \quad \text{for} \quad \mathbf{n}\cdot\mathbf{H}(z,\Xi_o) \quad ,$$

which is the component of $\mathbf{H}(z,\Xi_o)$ along the direction of the unit inward normal $\mathbf{n}$ to a unit area at depth z. $\mathbf{H}(z,\Xi_o)$ is the vector irradiance generated by radiant flux at z arriving from the general subregion Ξ_o of the unit sphere Ξ. If $\Xi_o = \Xi$, then $\mathbf{H}(z,\Xi_o) = \mathbf{H}(z)$ the usual vector irradiance at z. The quantity $\bar{H}(z,\mathbf{n},\Xi_o')$ is the associated (net) irradiance on the unit area contributed by the complement Ξ_o' of Ξ_o with respect to Ξ. Because of the assumed stratification, $\mathbf{H}(z,\Xi_o)$ (and hence all its components) depends only on z. By writing:

$$\text{"}\bar{H}_q(z,n,\Xi_o)\text{"} \quad \text{for} \quad \frac{b(z,\mathbf{n},\Xi_o')\,\bar{H}(z,\mathbf{n},\Xi_o')}{a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)} \quad , \tag{11}$$

we may write (10) as:

$$\frac{d\bar{H}(z,\mathbf{n},\Xi_o)}{dz} = -\left[a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)\right]\left[\bar{H}(z,\mathbf{n},\Xi_o) - \bar{H}_q(z,\mathbf{n},\Xi_o)\right] \tag{12}$$

so that:

(a) $a(z,\mathbf{n},\Xi_o) + b(z,\mathbf{n},\Xi_o)$ is the *attenuation function* for $\bar{H}(z,\mathbf{n},\Xi_o)$,

(b) $\bar{H}_q(z,\mathbf{n},\Xi_o)$ is the *equilibrium function* for $\bar{H}(z,\mathbf{n},\Xi_o)$.

The transport equation (10) is a generalization of the standard two-flow equations for $H(z,+)$ and $H(z,-)$ (Chapter 8). (In the latter case, for example, $\Xi_o = \Xi_-$, the downward hemisphere, and $\mathbf{n} = -\mathbf{k}$, where $\mathbf{k}$ is the unit outward normal to the plane-parallel medium.)

Each of the four preceding transport equations may be cast into a *canonical form* (see (2) of Sec. 11.2) by introducing the appropriate K - function for the associated radiometric quantity (see general definition (1) of Sec. 11.4). Therefore, a transport equation for each of these K - function exists, and is of the form (1) of Sec. 11.3. The equilibrium principle holds for N^n, N_*, $\mathbf{H}$, and U^n.

Summary and Conclusion

To summarize, the domain of applicability of the universal transport equation (1) of Sec. 11.3 is quite wide. In fact its domain covers the totality of radiometric functions used and known to date in radiative transfer theory (the 17 distinct types of radiometric concepts and their corresponding K - functions discussed above--34 concepts in all). By means of it, the general mathematical structure of the light field in plane-parallel media can be contained in a single unifying framework, and the necessity of invoking individual discussions and principles for each of the many radiometric quantities, at least on a logical level, is now obviated. All this from the interaction principle. Hence:

*Frustra fit per plura quod potest fieri per pauciora.**

William of Ockham (ca. 1300-1347)

11.5 Bibliographic Notes for Chapter 11

The concept of a universal radiative transport equation was introduced in [240]. The mathematical vehicle of the universal radiative transport equation is that of a Riccati differential equation in factored form:

$$\frac{df(x)}{dx} = \mu(x)\,[\delta f(x) - f_\alpha(x)]\,[f(x) - f_q(x)]$$

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*It will be futile to employ more [principles] when it is possible to employ fewer.

and with each radiometric quantity f (or K-function, or reflectance function, etc.) the equation associates two auxiliary functions f_α, f_q which, as was seen in the text, play the roles of attenuation and quilibrium functions, respectively. For an elementary discussion of the nondegenerate Riccati equation, see, e.g., [116]. For modern developments of the theory of nondegenerate Riccati equations pertinent to possible radiative transfer applications, see the work of Redheffer [254], [255], [256], [257], and also Reid [261] and [262]. These mathematical studies are also of potential applicability to the operator equations for the R and T operators in Chapter 7, as noted in Sec. 7.15. In view of the work of this chapter and that of Chapter 7, along with the results of the work of Redheffer and Reid, it is clear that the Riccati differential equation enters a productive new phase in mathematical physics.

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